Homoclinic Orbits near Heteroclinic Cycles with one Equilibrium and one Periodic Orbit

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Abstract

We analyze homoclinic orbits near codimension–1 and –2 heteroclinic cycles between an equilibrium and a periodic orbit for ordinary differential equations in three or higher dimensions. The main motivation for this study is a self-organized periodic replication process of travelling pulses which has been observed in reaction diffusion equations. We establish conditions for existence and uniqueness of countably infinite families of curve segments of 1–homoclinic orbits which *accumulate* at codimension–1 or –2 heteroclinic cycles. The main result shows the *bifurcation* of a number of curves of 1–homoclinic orbits from such codimension–2 heteroclinic cycles which depends on a winding number of the transverse set of heteroclinic points. In addition, a leading order expansion of the associated curves in parameter space is derived. Its coefficients are periodic with one frequency from the imaginary part of the leading stable Floquet exponents of the periodic orbit and one from the winding number.

Key words: homoclinic bifurcation, heteroclinic cycles, winding number, self-organization

1 Introduction

Heteroclinic connections in ordinary and partial differential equations can serve as models and explanations for interfaces and switching between two states. They frequently occur in the spatial dynamics of spatially one-dimensional partial differential equations and systems with symmetries, cf. e.g. [4,18].

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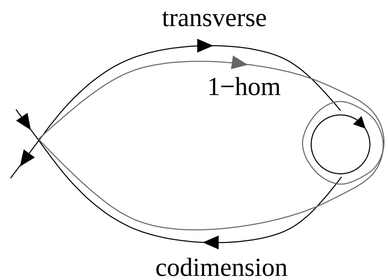

Fig. 1. Schematic picture of a typical heteroclinic cycle between an equilibrium and periodic orbit, and a nearby 1–homoclinic orbit. One connection is generically transverse; see text.

Heteroclinic networks, where heteroclinic connections are building blocks, allow to model more complicated switching behavior. A natural question is whether the building blocks of such a network form a skeleton for the nearby dynamics in phase space for close-by parameters. When all nodes in the network are hyperbolic fixed points, several questions of this type have been addressed and answered for generic or symmetric flows and diffeomorphisms, cf. e.g. [5,17,21,35].

In this article, bifurcation studies of heteroclinic networks are extended to heteroclinic cycles between one hyperbolic equilibrium p_0 and one maximally hyperbolic periodic orbit γ, see figure 1. We consider all cases where the sum of the codimensions of the two heteroclinic connections is 1 or 2. An important difference to heteroclinic cycles between equilibria is that the phase shift of a periodic orbit introduces a non hyperbolic direction which counts towards stable as well as unstable dimensions. For instance, this implies that not both heteroclinic orbits can have codimension 1, i.e. generically one is transverse.

To see this, let $i_\gamma = \dim(\mathcal{W}^{\mathrm{cu}}(\gamma))$ and $i_{p_0} = \dim(\mathcal{W}^{\mathrm{u}}(p_0))$ denote the unstable dimensions. In n ambient dimensions, the stable dimension of γ is $\dim(\mathcal{W}^{\mathrm{sc}}(\gamma)) = n - i_\gamma + 1$. The codimension of a heteroclinic orbit is at least as predicted by the number of dimensions in which the intersection of the relevant stable and unstable linear spaces can generically be destroyed: $n+1$ minus sum of stable and unstable dimension of target and source respectively, and zero if this number is negative. This may also be interpreted as the number of dimensions lacking for transversality. Codimension 1 from γ to p_0 means $n+1-(i_\gamma+n-i_{p_0}) = 1 \Leftrightarrow i_\gamma = i_{p_0}$, so for the other heteroclinic connection we obtain $(n - i_\gamma + 1) + i_{p_0} = n + 1$, hence it is generically transverse. Therefore firstly, in this sense a codimension–1 heteroclinic cycle is equivalent to $i_\gamma = i_{p_0}$ modulo time reversal. Secondly, a codimension–2 heteroclinic cycle necessarily involves a codimension–2 heteroclinic orbit, say from γ to p_0, and so the only possibility modulo time reversal is $i_\gamma + 1 = i_{p_0}$. Hence, for a heteroclinic cycle of codimension d for $d = 1, 2$, one orbit has codimension d while the other is transverse, and we note that the transverse set of heteroclinic points is typically d-dimensional.

In summary, we consider the following cases in this article and henceforth 'codimension d heteroclinic cycle' refers to these. For more details see hypotheses 2 and 3 in section 3.2.

Definition 1 *In a flow, a heteroclinic cycle between a hyperbolic equilibrium* p_0 *and a maximally hyperbolic periodic orbit* γ *is called codimension* d *for* $d = 1, 2$, *if* $i_{p_0} = i_\gamma + d - 1$.

Time reversal for $i_{p_0} = i_\gamma + d - 1$ implies that $i_{p_0} = i_\gamma - d$, and so our results also cover these cases. Concerning the ambient dimension n, we note that $n \geq 3$, because γ needs nontrivial stable and unstable manifolds. Moreover, $d = 2$ implies $n \geq 4$, because $n - i_{p_0} \geq 1$ and $i_{p_0} = i_\gamma + 1$ by Definition 1, so $i_\gamma \geq 2$ yields $n - i_{p_0} + i_{p_0} \geq 4$.

Heteroclinic networks involving periodic orbits have been considered previously for diffeomorphisms in terms of nonwandering sets and topology of bifurcation sets, cf. [11], and for three dimensional flows in [20]. Flows near homoclinic orbits to an equilibrium at a Hopf bifurcation and emerging heteroclinic orbits were studied in e.g. [10,16]. Numerically and analytically, heteroclinic cycles with periodic orbits have recently been found in several cases, cf. [4,14,22,23,36,37]. We point out analytic work concerning bifurcations for periodically forced systems in [41] and Hamiltonian systems near resonance in [14]. The splitting, not any bifurcation, of the codimension–2 heteroclinic connection in codimension–2 heteroclinic cycles with winding number 1 (see hypothesis 5) was shown in [28]. This also follows from our results. More generally, the codimension of heteroclinic connections between hyperbolic periodic orbits in infinite dimensions was studied in [13].

The main results reported in this article are valid for vector fields in any ambient dimension under generic and Melnikov-type conditions. For the bifurcation of smooth curves of homoclinic orbits an additional global topological condition is assumed. This appears to be the first result concerning codimension-2 homoclinic bifurcation from heteroclinic cycles with periodic orbits in flows. The technique is amenable to identifying any n–homoclinic or periodic orbit near such a heteroclinic cycle. We expect it applies to more general codimensions and networks, and possibly to infinite dimensions. Our setup is based on hyperbolicity and transversality. The backbone of the method are recurrent dynamics and uniform estimates near the periodic orbit, as well as exponential trichotomies. The latter are projections like exponential dichotomies, which in addition account for the center direction induced by the phase shift of the periodic orbit. The four main results presented in this article may be described as follows.

Theorem 1: Fix a pair of solutions which converge to a hyperbolic periodic orbit in forward and backward time respectively with the same asymptotic phase. All solutions in a neighborhood of these can be parametrized by a countably infinite family of smooth curves which wind around the periodic orbit j–times for any sufficiently large j. These solutions are exponentially close in j to the given pair of solutions.

Theorem 2: Near a typical codimension–1 or –2 heteroclinic cycle, there exists a unique, countably infinite family of smooth curve segments of 1–homoclinic orbits to the equilibrium. The 1–homoclinic orbits from the j–th connected curve wind around the periodic orbit $(j + j_0)$–times for some j_0 and are exponentially close to the cycle in j.

(A 1–homoclinic enters and leaves a suitable neighborhood of the periodic orbit only once.)

Theorem 3: From a typical codimension–2 cycle with finite winding number $m \geq 1$ (hypothesis 5), precisely m smooth curves of 1–homoclinic orbits to the equilibrium bifurcate.

Theorem 4: In the codimension–2 case and under hypothesis 5 with $1 \leq m < \infty$ the leading order expansion of the parameter curve has one frequency from the imaginary part of the leading stable Floquet exponents and one from the winding number. There is no additional condition on the spectral gaps.

The main technique to prove theorems 1 and 2 is an extension of Lin's method, cf. [21], which uses exponential di- and trichotomies for the variation about heteroclinic connections towards equilibrium and periodic orbit respectively. Major difficulties in this approach are the lack of hyperbolicity due to the phase shift of the periodic orbit, and the periodic distance of forward versus backward approach to the periodic orbit. The first corresponds to presence of essential spectrum in the case of spatial dynamics and is essentially overcome by using exponentially weighted spaces. The second difficulty generally excludes a connected curve of 1–homoclinic orbits which stays close to the heteroclinic cycle due to a phase coherence condition with respect to the periodic orbit. This is satisfied periodically and enumerates the countably infinite family of curves of 1–homoclinic orbits which accumulate at the heteroclinic cycle.

The problem of coherent phases does not appear for heteroclinic cycles between equilibria, and poses major challenges for the analysis presented here. These appear already on the level of theorem 1, which parametrizes all solutions close the heteroclinic cycles by fixed points of a certain solution operator derived from the variation of constants formula.

The main point concerning theorem 3 is that codimension–2 heteroclinic cycles have a winding number, which describes how often the transverse heteroclinic set winds around the periodic orbit and equilibrium. For instance this is 1 near a Hopf bifurcation, and 2 if the two-dimensional transverse heteroclinic set is a Möbius strip near the periodic orbit. On the other hand, for winding number *zero*, the phase coherence problem generally prevents a homoclinic bifurcation from the cycle. While the bifurcation of 1–homoclinic orbits may not be surprising, the influence of the global topology on existence and expansion suggests that bifurcation from such heteroclinic cycles may be rather complicated.

Since the length of 1–homoclinic orbits approaching the heteroclinic cycle diverges, we can interpret this bifurcation as a 'blue sky catastrophe' for homoclinic orbits. This notion has been introduced for periodic orbits as a codimension–1 bifurcation where both period and length diverge to infinity, cf. e.g. [34]. We expect that the periodic orbits which accompany these homoclinic orbits undergo a 'classical' blue sky catastrophe.

Concerning theorem 4 we remark that if both frequencies are nontrivial, then resonances conceivably prevent a spiraling curve, which is not possible for analogous heteroclinic cycles between equilibria. In addition, intersections with non-leading strong stable fibers may be structurally stable which alludes to the possibility of a countably infinite number of structurally stable bifurcations akin to orbit flip bifurcations. The expansion does not assume conditions on the unstable spectral gap at the periodic orbit, which, to the author's knowledge, has not been established previously for heteroclinic bifurcations.

Theorems 3 and 4 establish a correspondence of codimension–2 heteroclinic cycles between an equilibrium and periodic orbit with winding number 1 and analogous codimension–2 heteroclinic cycles between two equilibria called 'T-points' (for 'terminal points'), cf. [12]. For both types of heteroclinic cycles, suitable conditions cause the bifurcation of a locally unique curve of 1–homoclinic orbits of the parameter curves.

In addition, for spatial dynamics the geometry of the parameter curve is related to the socalled absolute spectrum of the periodic orbit or second equilibrium p_1, cf. [32,26]. In the case of two equilibria, neglecting technical details, if only one curve of the essential spectrum of p_1 is unstable and the absolute spectrum of p_1 contains the origin, then the path spirals to leading order, otherwise it is monotone. Leading order spiraling of the parameter curve actually allows to conclude an absolute instability for the partial differential equation

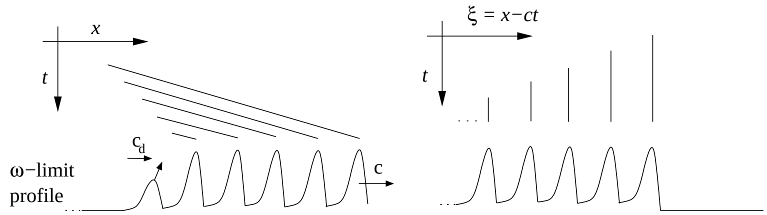

Fig. 2. Schematic space-time plot of tracefiring, in the original frame and comoving with the primary pulse with speed $c > 0$.

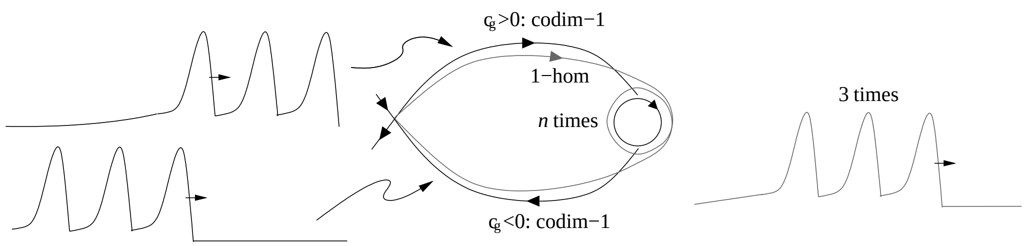

Fig. 3. Scheme of the heteroclinic cycle occurring in spatial dynamics of tracefiring.

from the finite dimensional travelling wave ODE, cf. [24,29,31]. In an absolute instability perturbations are not convected away, but grow pointwise, cf. [32].

Homoclinic bifurcations from heteroclinic cycles with a periodic orbit are generally different. Firstly, the global topology causes the bifurcation of several curves of 1–homoclinic orbits in a typical setting if the winding number is larger than one. Secondly, due to possible resonances of eigenvalues and the period of the periodic orbit, the parameter curves may be monotone to leading order despite an absolute instability involving the origin.

As an application, these results can partially explain the phenomenon of 'tracefiring', in particular for the three-component Oregonator model of the light-sensitive Belousov-Zhabotinskij reaction. The work presented in this article originated from analyzing tracefiring, see [26] for details. In tracefiring, a primary pulse loses stability and starts to replicate itself: secondary pulses periodically grow out the wake of the preceding travelling pulse-chain whose length is thereby incremented periodically, see figure 2. This self-organized process has two characteristic speeds: the speed of the primary pulse and the replication speed. This has been observed numerically in several other models [3,9,39,38], and named e.g. 'secondary trailing waves' in [38]. The patterns involved are also reminiscent of intermittency patterns for coupled cell dynamics found e.g. in [4].

In a comoving frame with the primary pulse's constant speed, stationary building blocks of tracefiring together constitute a heteroclinic cycle in the spatial direction ('spatial dynamics'), cf. figure 3. Now n–pulses correspond to 1–homoclinic orbits close to the heteroclinic cycle.

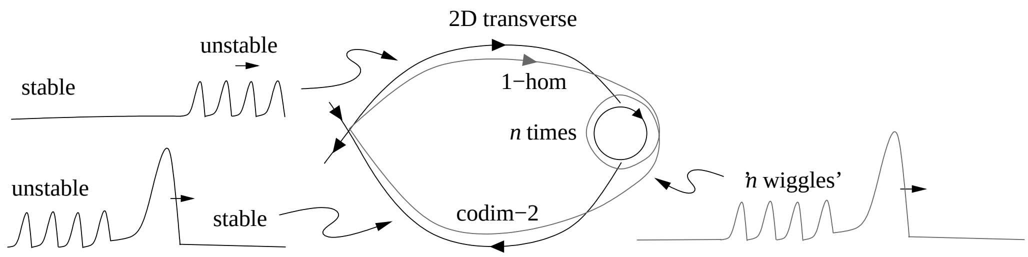

Fig. 4. Scheme of the heteroclinic cycle mediating the instability in the Oregonator.

The existence of these building blocks can be explained by the results in this article: the accumulation of homoclinic orbits at a codimension 1 heteroclinic cycle between the background state and the stationary periodic wave train. The results of [26] imply that a spatially heteroclinic cycle between a stable steady state and stable wave train is codimension–1 and determined by wave train's group velocity c_{g}, cf. figure 3.

For the Oregonator, the loss of stability of the primary pulse is mediated by pronounced oscillations in its wake. This may be interpreted as the presence of a nearby heteroclinic cycle with a periodic orbit reminiscent of the oscillations in the wake, cf. figure 4. Spectral computations show that this would be a codimension–2 heteroclinic cycle and the wave train in the wake is absolutely unstable, cf. [26]. The loss of stability can therefore be explained by closeness to the heteroclinic cycle. In this way, the aforementioned 'T-points' have been used to partially explain trace- and the related backfiring, cf. [1], in cases where a second *steady state* interacts with the pulse, [19,37,39].

This article is organized as follows. In section 2 the setting, notation and certain exponential trichotomies are introduced. Using these, we locate all 1–homoclinic orbits near the heteroclinic cycle in section 3. In section 4 we pathfollow these to prove theorem 3, and section 5 contains the derivation of a leading order expansion of the parameter curve for these 1–homoclinic orbits.

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2 Setting and exponential trichotomies

Notation *We use d to denote the codimension of the heteroclinic cycles between an equilibrium and periodic orbit. Unless stated otherwise, $d = 1$ or $d = 2$ are both valid.*

We consider the following ODE in $\mathbb{R}^n$ with $n \geq 2 + d$, $\mu \in \mathbb{R}^d$, and $\dot{} = \frac{\mathrm{d}}{\mathrm{d}\xi}$, f of class C^{k+1} in u and μ for $k \geq 1$.

$$\dot{u} = f(u; \mu) \tag{2.1}$$

We assume that for $\mu = 0$ equation (2.1) has an equilibrium p_0 and a periodic solution γ of minimal period $T_\gamma > 0$. Consider the first variation [2] about γ, $\dot{v} = \partial_u f(\gamma(\xi); 0)v$, with associated evolution operator $\Phi_\gamma(\xi, \zeta) : \mathbb{R}^n \to \mathbb{R}^n$, $\xi, \zeta \in \mathbb{R}$. By periodicity there is a Floquet representation $\Phi_\gamma(\xi, 0) = A_{\text{per}}(\xi)\mathrm{e}^{R\xi}$, where $A_{\text{per}}(\xi + T_\gamma) = A_{\text{per}}(\xi)$, $A_{\text{per}}(0) = \mathrm{Id}$ and R has a kernel, cf. [7]. We assume hyperbolicity, i.e. the zero eigenvalue of R is algebraically and geometrically simple and there are constants $\kappa_0 > 0$ and $\kappa > 0$ such that for $\operatorname{dist}(A, B) := \inf\{|a - b|, a \in A, b \in B\}$ it holds that

$$\operatorname{dist}(\operatorname{spec}(\partial_u f(p_0; 0)), \mathrm{i}\mathbb{R}) > \kappa_0 \quad \text{and} \quad \operatorname{dist}(\operatorname{spec}(R) \setminus \{0\}, \mathrm{i}\mathbb{R}) > \kappa. \tag{2.2}$$

Definition 2 (Morse indices) *The Morse index i_{p_0} of the equilibrium p_0 is the dimension of the unstable manifold $\mathcal{W}^{\mathrm{u}}(p_0)$. The Morse index i_γ of the periodic orbit γ is the dimension of the center-unstable manifold $\mathcal{W}^{\mathrm{cu}}(\gamma)$.*

Any solution u to (2.1) that converges to γ in forward time has an asymptotic phase $\alpha \in [0, T_\gamma)$: $u(\xi) - \gamma(\xi + \alpha) \to 0$ as $\xi \to \infty$, cf. [7]. The collection of all points in the stable manifold with the same asymptotic phase α is the strong stable fiber $\mathcal{W}^{\mathrm{ss}}_\alpha(\gamma)$ and the stable manifold is $\mathcal{W}^{\mathrm{sc}}(\gamma) = \cup_{\alpha \in [0, T_\gamma)} \mathcal{W}^{\mathrm{ss}}_\alpha(\gamma)$, cf. e.g. Theorem 2.2, Chapter 13 in [7].

Using a suitable Euler multiplier and local coordinate change γ is the locally unique hyperbolic periodic orbit of $f(\cdot; \mu)$ with hyperbolicity rate at least κ and Morse index $i(\gamma)$ for all $\mu \in \Lambda \subset \mathbb{R}^d$, where Λ is a neighborhood of zero, cf. e.g. [28].

We assume there are heteroclinic orbits $q_1(\xi)$, $q_2(\xi)$ with asymptotic phase 0, which connect p_0 to γ and γ to p_0 respectively. The ambient dimension is at least three, because the periodic orbit needs nontrivial stable and unstable manifolds. Since any orbit converging to γ has an asymptotic phase in $[0, T_\gamma)$, $\alpha = 0$ can be achieved by considering $q_1(\cdot - \alpha)$ instead of q_1. The assumption on

[2] The letter v will be used to denote different objects.

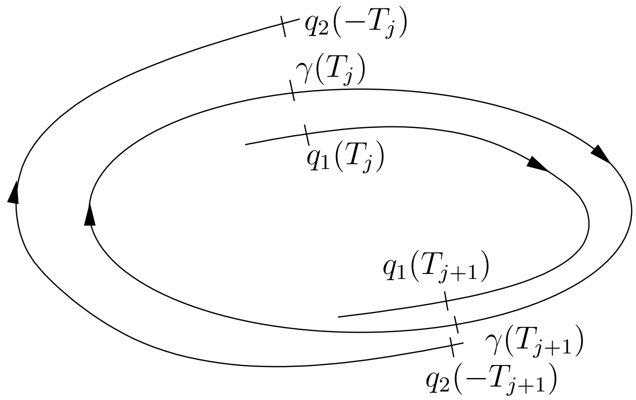

Fig. 5. The asymptotically $(T_\gamma/2)$–periodic distance of $q_1(L)$ and $q_2(-L)$.

the zero asymptotic phase of q_2 when approaching γ in backward time can be achieved independently by a suitable choice of $\gamma(0)$ or $q_2(0)$. The role of α will become clear in section 4, where heteroclinic orbits in $\mathcal{W}_1 := \mathcal{W}^{\mathrm{u}}(p_0) \cap \mathcal{W}^{\mathrm{cs}}(\gamma)$ are varied.

Hypothesis 1 *At $\mu = 0$ the flow of equation (2.1) possesses a hyperbolic equilibrium p_0 and a hyperbolic periodic orbit γ with rates of hyperbolicity κ_0 and κ as in (2.2). There are heteroclinic orbits $q_1(\xi)$, $q_2(\xi)$, which connect p_0 to γ and γ to p_0 respectively and both have asymptotic phase* 0 *with respect to γ. The heteroclinic cycle consisting of p_0, γ, q_1 and q_2 is codimension–d.*

Under the hyperbolicity assumption, theorem 4.3 in chapter 13 of [7] implies that there is a constant $K_0 > 0$ such that ($\alpha = 0$ under hypothesis 1)

$$\begin{aligned} |q_1(\xi) - \gamma(\xi + \alpha)| &\le K_0 \mathrm{e}^{-\kappa\xi} \text{ for } \xi \ge 0 \\ |q_2(\xi) - \gamma(\xi)| &\le K_0 \mathrm{e}^{\kappa\xi} \text{ for } \xi \le 0 \\ |q_1(\xi) - p_0| &\le K_0 \mathrm{e}^{\kappa_0\xi} \text{ for } \xi \le 0 \\ |q_2(\xi) - p_0| &\le K_0 \mathrm{e}^{-\kappa_0\xi} \text{ for } \xi \ge 0. \end{aligned} \tag{2.3}$$

Notice $\gamma(T_\gamma/2) = \gamma(-T_\gamma/2)$, so the heteroclinic orbits to γ and their asymptotic phases in (2.3) give rise to an unbounded sequence $T_j := jT_\gamma/2$ so that

$$|q_1(T_j) - q_2(-T_j)| \le C\mathrm{e}^{-\kappa T_j}. \tag{2.4}$$

This expresses the asymptotically periodic distance along the heteroclinic orbits towards γ, cf. figure 5, and $2j$ counts how often $q_1(\xi)$ and $q_2(\xi)$ wind around γ for $\xi \in [0, T_j]$.

Notation *We use the following convention about constants that depend on $f(\cdot; 0)$, but not on μ or L in estimates: C denotes constants, which in between steps of a computation may absorb multiplicative factors or take a maximum* *value of finitely many constants. This is implicitly done to serve readability by focussing on the essential asymptotic analysis. The main ingredients of these constants are denoted K_j for integers j. Constants C_j with a subscript j are fixed constants.*

For a solution u to (2.1), the variation $w_j = u - q_j$ about the heteroclinic orbit q_j, for $j = 1, 2$, (see figure 6) satisfies

$$\begin{aligned} \dot{w}_j &= f(q_j(\cdot) + w_j; \mu) - f(q_j(\cdot); 0) = A_j(\cdot) w_j + g_j(w_j, \cdot; \mu) \\ A_j(\xi) &:= \partial_u f(q_j(\xi); 0) \\ g_j(w_j, \xi; \mu) &:= f(q_j(\xi) + w_j; \mu) - f(q_j(\xi); 0) - A_j(\xi) w_j. \end{aligned} \tag{2.5}$$

From the definitions of g_j and the independence of γ on μ, we conclude that there is a constant K_1 so that for all ξ and w_1, w_2 in a neighborhood of zero

$$\begin{aligned} |g_j(w_j, \xi; \mu)| &\leq K_1(|w_j|^2 + |\mu|(|w_j| + |q_j(\xi) - \gamma(\xi)|)) \\ |\partial_{w_j} g_j(w_j, \xi; \mu)| &\leq K_1(|w_2| + |\mu|). \end{aligned} \tag{2.6}$$

Let $\Phi_j(\xi, \zeta)$, $j = 1, 2$, be the solution operators (evolutions) to

$$\dot{v} = A_j(\cdot) v.$$

Notation *In the following, slashes '/' separate alternative, valid choices.*

Definition 3 *Let $\phi(\xi, \zeta)$ be the evolution operator of a linear non-autonomous ODE $\dot{u} = A(\xi) u$. We say $\phi(\xi, \zeta)$ has an* ***exponential trichotomy*** *on $I = \mathbb{R}$, $I = \mathbb{R}^+$ or $I = \mathbb{R}^-$, if there exist families of complementary projections $P^{\mathrm{s}}(\xi)$, $P^{\mathrm{c}}(\xi)$ and $P^{\mathrm{u}}(\xi)$, i.e. $P^{\mathrm{s}} + P^{\mathrm{c}} + P^{\mathrm{u}} \equiv \mathrm{Id}$ and $P^{\mathrm{s}}(P^{\mathrm{c}} + P^{\mathrm{u}}) \equiv 0$, $P^{\mathrm{c}}(P^{\mathrm{s}} + P^{\mathrm{u}}) \equiv 0$, $P^{\mathrm{u}}(P^{\mathrm{s}} + P^{\mathrm{c}}) \equiv 0$, which are continuous for $\xi \in I$, and there exist constants $K_2 > 0$, $-\kappa^s < 0 < \kappa^u$, such that for $\xi, \zeta \in I$, $u \in \mathbb{R}^n$*

$$\begin{aligned} |\phi(\xi, \zeta) P^{\mathrm{s}}(\zeta) u| &\leq K_2 \mathrm{e}^{-\kappa^s(\xi - \zeta)} |u|, \quad \xi \geq \zeta \\ |\phi(\xi, \zeta) P^{\mathrm{u}}(\zeta) u| &\leq K_2 \mathrm{e}^{\kappa^u(\xi - \zeta)} |u|, \quad \xi \leq \zeta \\ |\phi(\xi, \zeta) P^{\mathrm{c}}(\zeta) u| &\leq K_2 |u|, \quad \forall \xi, \zeta \\ \phi(\xi, \zeta) P^{\mathrm{s/c/u}}(\zeta) &= P^{\mathrm{s/c/u}}(\xi) \phi(\xi, \zeta), \ \xi \geq \zeta \text{ or } \xi \leq \zeta \text{ as above.} \end{aligned} \tag{2.7}$$

We call P^{s} the stable, P^{c} the center, and P^{u} the unstable projection, and their images $E^{\mathrm{s/c/u}}(\xi) := \mathrm{Rg}(P^{\mathrm{s/c/u}}(\xi))$ stable, center and unstable spaces. The ξ-independent dimension $\dim(\ker((P^{\mathrm{u}} + P^{\mathrm{c}})(\xi)))$ is referred to as the Morse index of the exponential trichotomy.

If $P^{\mathrm{c}} \equiv 0$ the exponential trichotomy is called an ***exponential dichotomy****.*

The estimate for the center direction is cast for our applications, where the center space is always one-dimensional. For more general purposes the center estimate K_2 is replaced by $\tilde{K}_2 e^{\tilde{\eta}|\xi|}$, $0 < \tilde{\eta} \leq \kappa^{\mathrm{u}}, \kappa^{\mathrm{s}}$, see e.g. [33]. While the crucial estimates and statements we will derive also hold for this definition, the above formulation suffices and is more convenient.

Since the trichotomy projections are complementary, we obtain $P^{\mathrm{sc}} := P^{\mathrm{s}} + P^{\mathrm{c}}$ and P^{u} as complementary families of projections as well as $P^{\mathrm{cu}} := P^{\mathrm{s}} + P^{\mathrm{c}}$ and P^{s}. We define center-stable and center-unstable families of spaces $E^{\mathrm{sc}}(\xi) := \mathrm{Rg}(P^{\mathrm{sc}}(\xi))$, and $E^{\mathrm{cu}}(\xi) := \mathrm{Rg}(P^{\mathrm{cu}}(\xi))$.

Remark 1 *The main example of a trichotomy on $\mathbb{R}$ for our purposes occurs in $\dot{v} = \partial_u f(\gamma(\xi); 0)v$. By the hyperbolicity assumption, the center space is one-dimensional and the trichotomy estimates follow from the spectral assumptions on R of the aforementioned Floquet representation of the evolution $\Phi_\gamma(\xi, 0) = A_{\mathrm{per}}(\xi)\mathrm{e}^{R\xi}$. Let $P_\gamma^{\mathrm{s/c/u}}(\xi)$ be the stable/center/unstable projections for this trichotomy. We further conclude that the images of the stable and unstable projections are the stable and unstable eigenspaces $E_R^{\mathrm{s/u}}$ of R transported with $A_{\mathrm{per}}(\xi)$, because $E_\gamma^{\mathrm{s/u}}(\xi) = A_{\mathrm{per}}(\xi)E_R^{\mathrm{s/u}}$. Since $\dot{\gamma}$ is a periodic solution and* $\ker(R)$ *one-dimensional we have $E_\gamma^{\mathrm{c}}(\xi) = \mathrm{span}\{\dot{\gamma}(\xi)\}$.*

Lemma 1 (and notation) *Under hypothesis 1 the evolution $\Phi_1(\xi, \zeta)$ possesses an exponential dichotomy on $\mathbb{R}^-$ and $\Phi_2(\xi, \zeta)$ on $\mathbb{R}^+$ with stable and unstable rates at least κ_0. We denote the stable and unstable projections of these by $P_{-1}^{\mathrm{s}}(\xi)$, $P_{-1}^{\mathrm{u}}(\xi)$, $\xi \leq 0$, for Φ_1 and $P_{+2}^{\mathrm{s}}(\xi)$, $P_{+2}^{\mathrm{u}}(\xi)$, $\xi \geq 0$, for Φ_2.*

Furthermore, $\Phi_1(\xi, \zeta)$ possesses an exponential trichotomy on $\mathbb{R}^+$ and $\Phi_2(\xi, \zeta)$ on $\mathbb{R}^-$. We denote the stable/center/unstable projections for Φ_1 by $P_{+1}^{\mathrm{s/c/u}}(\xi)$, $\xi \geq 0$, and for Φ_2 by $P_{-2}^{\mathrm{s/c/u}}(\xi)$, $\xi \leq 0$. The projections' images $E^{\mathrm{s/c/u}}(\xi)$ inherit the sub-indices of the projections and

$$
\begin{aligned}
E_{+1}^{\mathrm{c}}(\xi) &= \mathrm{Rg}(P_{+1}^{\mathrm{c}}(\xi)) = \mathrm{span}\{\dot{q}_1(\xi)\},\ \xi \geq 0\ , \\
E_{-2}^{\mathrm{c}}(\xi) &= \mathrm{Rg}(P_{-2}^{\mathrm{c}}(\xi)) = \mathrm{span}\{\dot{q}_2(\xi)\},\ \xi \leq 0\ .
\end{aligned}
$$

Proof. Generally, if $A(\xi)$ is constant and hyperbolic as $\xi \to \infty$ then $\dot{u} = A(\xi)u$ has an exponential dichotomy on $\mathbb{R}^+$, cf. e.g. [8]. This implies the claim about exponential dichotomies, because by hypothesis 1 $\lim_{\xi\to\infty} A_2(\xi) = \lim_{\xi\to-\infty} A_1(\xi) = \partial_u f(p_0; 0)$ is hyperbolic with rate κ_0.

However, the other asymptotic state $\partial_u f(\gamma(\xi); 0)$ is periodic with an algebraically and geometrically simple vanishing Floquet exponent in the direction of $\dot{\gamma}$. In this case, $A_j(\xi) - \partial_u f(\gamma(\xi); 0) = O(\mathrm{e}^{-\kappa|\xi|})$ for $\xi \to (-1)^j\infty$ implies that the perturbation is integrable. Therefore, the trichotomies follow from remark 1 and e.g. a slight modification of proposition 2 and 3 on p.35 in [8].

The expressions for the center-spaces follow, because these are one-dimensional, $\dot{q}_j(\xi)$, $j=1,2$ solve the respective variational equations and satisfy the center estimate. $\square$

The next lemma establishes projections that couple those of $\Phi_j(\xi,\zeta)$, $j=1,2$ near γ. We say that a collection of projections is complementary, if the image of any one projection lies in the kernel of the others, and the sum of the projections is the identity.

Lemma 2 *Assume hypothesis 1 and consider $P_{+1}^{\mathrm{s/u/c}}(L)$, and $P_{-2}^{\mathrm{s/u/c}}(-L)$ from lemma 1. There exists strictly positive constants $\epsilon_0 < T_\gamma/2$, K_3, L_0 such that the following holds. For $L \geq L_0$, with $|L-T_j| \leq \epsilon_0$ for some j, there exist complementary projections $P_L^{\mathrm{s/u/c}}$, continuous in L, whose images satisfy $\mathrm{Rg}(P_L^{\mathrm{s}}) = E_{-2}^{\mathrm{s}}(-L)$, $\mathrm{Rg}(P_L^{\mathrm{u}}) = E_{+1}^{\mathrm{u}}(L)$, $\mathrm{Rg}(P_L^{\mathrm{c}}) = E_{-2}^{\mathrm{c}}(-L) = \mathrm{span}\{\dot{q}_2(-L)\}$, and $|P_L^{\mathrm{s/u/c}}| \leq K_3$. In particular*

$$P_L^{\mathrm{s}} + P_L^{\mathrm{c}} + P_L^{\mathrm{u}} = \mathrm{Id}. \tag{2.8}$$

Proof. To obtain the desired uniformly bounded projections, we can apply lemma 7 from [40] to $P_{-2}^{\mathrm{sc}}(\xi)$ and $P_{+1}^{\mathrm{u}}(\xi)$, which converge to the complementary projections $P_\gamma^{\mathrm{sc}}(\xi)$ for $\xi \to -\infty$ and $P_\gamma^{\mathrm{u}}(\xi)$ for $\xi \to \infty$ respectively. The proof in [40] does not use the fact that the matrices involved are asymptotically constant and applies with minor modifications here as long as L is sufficiently close to the sequence T_j, i.e. there is j such that $|L-T_j| \leq \epsilon_0$ for some $\epsilon_0 > 0$. We obtain a pair of complementary projections $P_-^{\mathrm{s/u}}(L)$, as smooth in L as the original projections, and which satisfy

$$\mathrm{Rg}(P_-^{\mathrm{s}}(L)) = E_{-2}^{\mathrm{sc}}(-L) \quad , \quad \mathrm{Rg}(P_-^{\mathrm{u}}(L)) = E_{+1}^{\mathrm{u}}(L).$$

Using these and the center projection $P_{-2}^{\mathrm{c}}(-L)$ for the variation about q_2, we define the following projections

$$\begin{aligned} P_L^{\mathrm{sc}} &:= P_-^{\mathrm{s}}(L) \quad , \; P_L^{\mathrm{c}} := P_{-2}^{\mathrm{c}}(-L) P_L^{\mathrm{sc}} \\ P_L^{\mathrm{s}} &:= P_L^{\mathrm{sc}} - P_L^{\mathrm{c}} \; , \; P_L^{\mathrm{u}} := P_-^{\mathrm{u}}(L) \end{aligned}$$

We first verify the ranges. By definition $\mathrm{Rg}(P_L^{\mathrm{u}}) = E_{+1}^{\mathrm{u}}(L)$, while $\mathrm{Rg} P_L^{\mathrm{c}} = \mathrm{span}\{\dot{q}_2(-L)\}$ and $\mathrm{Rg} P_L^{\mathrm{s}} = E_{-2}^{\mathrm{s}}(-L)$ follow from

$$\begin{aligned} \mathrm{Rg}(P_L^{\mathrm{sc}}) &= E_{-2}^{\mathrm{sc}}(-L) = E_{-2}^{\mathrm{s}}(-L) \oplus \mathrm{span}\{\dot{q}_2(-L)\} \\ \mathrm{Rg}(P_{-2}^{\mathrm{c}}(L)) &= \mathrm{span}\{\dot{q}_2(-L)\}. \end{aligned}$$

We conclude from the complementarity that $P_L^{\mathrm{s}} + P_L^{\mathrm{c}} + P_L^{\mathrm{u}} = \mathrm{Id}$.

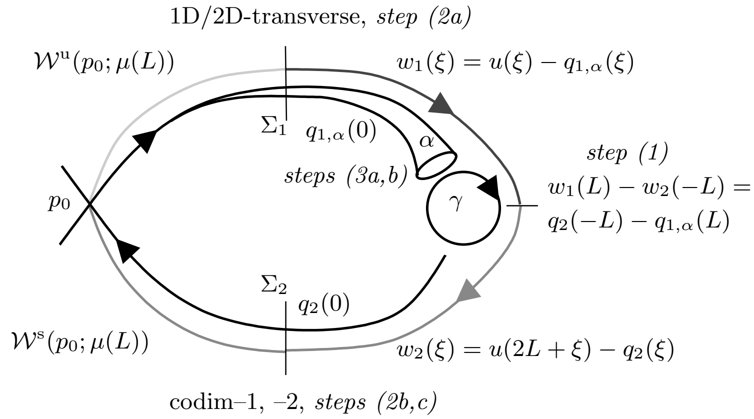

Fig. 6. Schematic picture of adapted Lin's method to find 1–homoclinic orbits near the heteroclinic cycle with a periodic orbit. Numbers indicate the steps describe in the text. For the codimension–2 case α parametrizes distinct heteroclinic orbits. For L see remark 3.

As to the kernels, the definitions give $\ker(P^{\mathrm{u}}_L)=\mathrm{Rg}(P^{\mathrm{sc}}_L)$, $\mathrm{Rg}(P^{\mathrm{u}}_L)=\ker(P^{\mathrm{sc}}_L)\subset\ker(P^{\mathrm{c}}_L)$, the above implies $\mathrm{Rg}(P^{\mathrm{s}}_L)\subset\ker P^{\mathrm{c}}_L$, and $\mathrm{Rg}(P^{\mathrm{c}}_{-2}(L)\subset\mathrm{Rg}(P^{\mathrm{sc}}_L)$ so

$$\begin{aligned}\ker(P^{\mathrm{c}}_L)&=\mathrm{Rg}(P^{\mathrm{u}}_L)+\mathrm{Rg}(P^{\mathrm{s}}_L)\\ \ker(P^{\mathrm{s}}_L)&=\ker(P^{\mathrm{sc}}_L-P^{\mathrm{c}}_L)=\mathrm{Rg}(P^{\mathrm{c}}_L)+\mathrm{Rg}(P^{\mathrm{u}}_L).\end{aligned}$$

□

Remark 2 *Let $L\geq L_0$, $|L-T_j|\leq\epsilon_0$ for some j. Notice that $E^{\mathrm{c}}_{+1}(L)$ is not necessarily contained in a specific kernel of the projections $P^{\mathrm{s/c/u}}_L$. But $\dot q_1(L)$ is exponentially close to $\dot\gamma(L)$, while $q_2(-L)$ is exponentially close to $\dot\gamma(-L)$. So in terms of suitable unit basis vectors $E^{\mathrm{c}}_{-2}(-L)$ and $\mathrm{Rg}(P^{\mathrm{c}}_L)$ are exponentially close to $\mathrm{span}\{\dot\gamma(-L)\}$ and $\mathrm{Rg}(P^{\mathrm{c}}_\gamma(-L))$ respectively. The spaces $E^{\mathrm{c}}_{+1}(T_j)$ and $E^{\mathrm{c}}_{-2}(-T_j)$ are exponentially close to $\mathrm{span}\{\dot\gamma(T_j)\}$ in this sense. In particular, for $L=T_j+O(\mathrm{e}^{-\kappa L})$ we have $P^{\mathrm{c}}_LP^{\mathrm{u}}_{-2}(-L)=O(\mathrm{e}^{-\kappa L})$, and $P^{\mathrm{c}}_LP^{\mathrm{s/u}}_{+1}(L)=O(\mathrm{e}^{-\kappa L})$, $P^{\mathrm{s/u}}_{+1}(L)P^{\mathrm{c}}_L=O(\mathrm{e}^{-\kappa L})$.*

3 1–homoclinic orbits near heteroclinic cycles

Using the variations (2.5) about the heteroclinic orbits, we next set up a fixed point formulation akin to Lin's method [21] to find 1–homoclinic orbits near the heteroclinic cycle in parameter and phase space.

The method to prove theorem 3 consists of six steps, cf. figure 6:

1 Parametrize all solutions passing near γ and q_1, q_2 (this proves theorem 1), i.e. 'glue' variations $w_1(L)$, $w_2(-L)$ and $q_1(L)$, $q_2(-L)$ together to obtain continuous solutions to (2.1).

2a Match the glued solutions with $\mathcal{W}^{\mathrm{u}}(p_0)$ near the transverse intersection $\mathcal{W}_1 = \mathcal{W}^{\mathrm{u}}(p_0) \cap \mathcal{W}^{\mathrm{cu}}(\gamma)$.

2b Match the transverse part by Ljapunov-Schmidt reduction near the intersection of $\mathcal{W}^{\mathrm{s}}(p_0)$ and $\mathcal{W}^{\mathrm{cs}}(\gamma)$.

2c Match the remaining part by Melnikov's method (this proves theorem 2) to obtain a family of curves of 1–homoclinic orbits.

3a Patch part of the curves of 1–homoclinic orbits together by varying the underlying heteroclinic for $d = 2$.

3b Patch all 1–homoclinic orbits together using the global topological hypothesis 5 to obtain a smooth connected curve of 1–homoclinic orbits and parameters that bifurcate from the heteroclinic cycle.

Remark 3 *For the codimension–2 case, let Γ denote a curve of heteroclinic points transverse to the flow. For the codimension–1 case set $\Gamma := \{q_1(0)\}$. In figure 6, the parameter L geometrically means the semi travel time between some small transverse sections Σ_1 and Σ_2 near Γ and $q_2(0)$ respectively: any solution u passing near the heteroclinic cycle has unique 'hit' times ξ_1, ξ_2 at which $u(\xi_1) \in \Sigma_1$ and $u(\xi_2) \in \Sigma_2$. The uniquely defined semi-travel-time from Σ_1 to Σ_2 is then $L = (\xi_2 - \xi_1)/2$. However, in the approach below, we need more flexibility and do not interpret L as this strict semi travel time. We will show later that L is close to the above semi travel time; for homoclinic solutions exponentially close, see theorem 2.*

Steps 1-2c involve successive elimination of variables by the following slightly non-standard, but well known, implicit function theorem. In lack of a reference, we prove it here in a unified form to be conveniently used for all the steps.

Proposition 1 *Let X, Y and Z be open neighborhoods of zero, and I an open set, all in some ambient Banach spaces. For convenience, we denote all norms by $|\cdot|$. Let $Q : Z \times I \to \mathcal{L}(Y, X)$, $R : Y \times Z \times I \to X$ and $S : I \to X$ be C^k, $k \geq 1$, in all variables. Assume that for $z \in Z$, $L \in I$ the linear map $Q(z, L)$ invertible and there exists positive constants C_1, C_2 such that $||Q^{-1}(z, L)|| \leq C_1$ and*

$$|R(y, z, L)| \leq C_2((|z| + |y|)|y| + |z|). \tag{3.1}$$

Let $C_* := C_1 C_2$, $r_y := \frac{4C_*+1}{4C_*(2C_*+1)}$, $r_z := \frac{1}{4C_*(2C_*+1)}$. Then any solution to

$$Q(z, L)y + R(y, z, L) = S(L) \tag{3.2}$$

for $y \in Y_* := \{y \in Y \,|\, |y| < r_y\}$, $z \in Z_* := \{z \in Z \,|\, |z| < r_z\}$ and $L \in I$ satisfies

$$|y| \leq 2(C_*|z| + C_1|S(L)|). \tag{3.3}$$

If in addition $C_1|S(L)| \le \frac{1}{8C_*}$ *for* $L \in I$ *and*

$$|\partial_y R(y,z,L)| \le C_2(|z|+|y|), \tag{3.4}$$

then there exists a C^k *function* $y^* : Z_* \times I \to Y_*$ *such that* $y^*(z,L)$ *uniquely solves (3.2) for* $y \in Y_*$.

Proof. By assumption on Q, we can rewrite the equation to be solved as

$$y = r(y,z,L) := -Q(z,L)^{-1}(R(y,z,L)+S(L)) \tag{3.5}$$

with a C^k function $r : Y \times Z \times I \to Y$, and for $y \in Y_*$ and $z \in Z_*$ we have by assumption $C_*(|z|+|y|) \le \frac{1}{2}$. Since $||Q^{-1}(z,L)|| \le C_1$ it follows from (3.1) that (3.5) implies

$$\begin{aligned} |y| &\le \tfrac{1}{2}|y| + C_*|z| + C_1|S(L)| \\ \Rightarrow |y| &\le 2(C_*|z| + C_1|S(L)|). \end{aligned} \tag{3.6}$$

By assumption (3.4) $r(\cdot,z,L)$ is a uniform contraction of Y_* for $z \in Z_*$, $L \in I$:

$$|r(y_1,z,L) - r(y_2,z,L)| \le \sup_{y\in Y_*} |Q(z,L)^{-1}(\partial_y R(y,z,L))||y_1-y_2| \le \frac{1}{2}|y_1-y_2|.$$

Finally, $C_1|S(L)| \le \frac{1}{8C_*}$ for $L \in I$ implies $r(\cdot,z,L)$ maps Y_* into itself for all $z \in Z_*$ and $L \in I$, because it follows from (3.6) and $|z| < r_z$ that

$$|r(y,z,L)| \le \frac{1}{2}|y| + C_*|z| + C_1|S(L)|) < \frac{1}{2}r_y + \frac{1}{4(2C_*+1)} + \frac{1}{8C_*} = r_y$$

Hence, the uniform contraction principle, e.g. [6] theorem 2.2, provides the locally unique fixed point $y^* \in Y_*$, which is C^k in $z \in Z_*$ and $L \in I$. □

3.1 Glued solutions

In this step, we only use the local structure near the periodic orbit. The global heteroclinic structure plays no role, and for any pair of solutions, one converging forward and one backward to a hyperbolic periodic orbit, we parametrize all orbits passing the periodic orbit near this solution pair using the approximate semi travel time L.

For $L > L_0$, we consider solutions $w_1(\xi_+)$, $\xi_+ \in [0,L]$, and $w_2(\xi_-)$, $\xi_- \in [-L,0]$, to (2.5), and use the projections from lemma 1 to denote $w_{+1}^{\mathrm{s/c/u}}(\xi_+) := P_{+1}^{\mathrm{s/c/u}}(\xi_+)w_1(\xi_+)$, $w_{-2}^{\mathrm{s/c/u}}(\xi_-) := P_{-2}^{\mathrm{s/c/u}}(\xi_-)w_2(\xi_-)$, as well as

$$W\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} := \begin{pmatrix} w_1(\xi_+)\\ w_2(\xi_-)\end{pmatrix} = \begin{pmatrix} w_{+1}^{\mathrm{s}}(\xi_+) + w_{+1}^{\mathrm{u}}(\xi_+) + w_{+1}^{\mathrm{c}}(\xi_+)\\ w_{-2}^{\mathrm{s}}(\xi_-) + w_{-2}^{\mathrm{u}}(\xi_-) + w_{-2}^{\mathrm{c}}(\xi_-)\end{pmatrix} \tag{3.7}$$

(We write vectors as columns or rows interchangeably.) The glued solutions will be concatenations of $w_1(\xi_+) + q_1(\xi_+)$ and $w_2(\xi_-) + q_2(\xi_-)$ on $[0, 2L]$ by means of

$$u(\xi) = \begin{cases} q_1(\xi) + w_1(\xi)\,,\ \xi \in [0, L] \\ q_2(\xi - 2L) + w_2(\xi - 2L)\,,\ \xi \in [L, 2L] \end{cases} \tag{3.8}$$

Hence, the variational parts have to satisfy the boundary condition $w_1(L) + q_1(L) = w_2(-L) + q_2(-L)$. We call variational solutions w_1, w_2 *glued*, if

$$b_L := q_2(-L) - q_1(L) = w_1(L) - w_2(-L). \tag{3.9}$$

For the contraction argument to find glued solutions, we will need b_L to be small. Since b_L is asymptotically periodic, we cannot expect to be able to track glued solutions for all L by our method without further assumptions.

Lemma 3 *Assume hypothesis 1 and take ϵ_0 from lemma 2. There exists a sequence of disjoint open intervals I_j for a $j_0 \geq 0$ with length $|I_j| \leq \mathrm{e}^{-\kappa \sup I_j} \leq \epsilon_0$ such that if $L \in I_j$ for some $j \geq 0$, then*

$$|b_L| \leq K_4 \mathrm{e}^{-\kappa \sup I_j}.$$

Proof. As noted in (2.4), for the period T_γ of γ we have that $\gamma(T_\gamma/2) - \gamma(-T_\gamma/2) = 0$. Hypothesis 1 implies that for $\xi \geq 0$ we have

$$|q_1(\xi) - \gamma(\xi)| \leq K_0 \mathrm{e}^{-\kappa\xi}$$
$$|q_2(-\xi) - \gamma(-\xi)| \leq K_0 \mathrm{e}^{-\kappa\xi}.$$

Hence, for $T_j := jT_\gamma/2$ it holds that $|b(T_j)| = |q_1(T_j) - q_2(-T_j)| \leq C\mathrm{e}^{-\kappa T_j}$. Define $I_j := (T_j - \mathrm{e}^{-\kappa(T_j+1)}/2, T_j + \mathrm{e}^{-\kappa(T_j+1)}/2)$, so the lengths satisfy $|I_j| \leq \exp(-\kappa \sup I_j)$. For $\bar{K}_4 := \sup_{\xi\in\mathbb{R}^+} |q_1'(\xi)| + \sup_{\xi\in\mathbb{R}^-} |q_2'(\xi)|$ and $L \in I_j$ we have

$$|q_1(L) - q_1(T_j)| + |q_2(-L) - q_2(T_j)| \leq \sup_{\xi\in I_j} |q_1'(\xi)|\,|L - T_j| \leq \bar{K}_4 |I_j|.$$

Thus, if $L \in I_j$ for some $j \geq 0$ then it holds that

$$\begin{aligned} |b_L| &= |q_1(L) - q_2(-L)| \\ &\leq |q_1(L) - q_1(T_j)| + |q_1(T_j) - q_2(T_j)| + |q_2(T_j) - q_2(L)| \\ &\leq \bar{K}_4|I_j| + 2K_0\mathrm{e}^{-\kappa T_j} \leq (\bar{K}_4 + 2K_0\mathrm{e}^{\kappa T_\gamma/2})\mathrm{e}^{-\kappa \sup I_j} \ \leq\ C\mathrm{e}^{-\kappa \sup I_j}. \end{aligned}$$

There is $j_0 \geq 0$ so that for $j \geq j_0$ the intervals I_j are all disjoint, because $|I_j| \to 0$. ◻

Notation $I_b := \cup_{j\geq j_0} I_j \subset \mathbb{R}^+$ *is the set of almost phase coherent L.* The periodic orbit's phase shift introduces lack of hyperbolicity, which causes some difficulties to obtain uniform estimates in L below. However, analogues of these can be achieved in exponentially weighted norms. It turns out that a good choice of spaces for the fixed point formulation of glued solutions are the products $\mathcal{X}^\eta_{\tilde{L}}$ of exponentially weighted spaces $C_{\eta,\tilde{L}} = C^0([0,\tilde{L}],\mathbb{R}^n)$ and $C_{-\eta,\tilde{L}} = C^0([-\tilde{L},0],\mathbb{R}^n)$ with norms

$$\begin{aligned}||w_1||_{\eta,\tilde{L}} &:= \sup\nolimits_{0\le\xi\le\tilde{L}} |\mathrm{e}^{\eta\xi} w_1(\xi)|\\ ||w_2||_{-\eta,\tilde{L}} &:= \sup\nolimits_{-\tilde{L}\le\xi\le 0} |\mathrm{e}^{-\eta\xi} w_2(\xi)|.\end{aligned}$$

With $W = \binom{w_1}{w_2}$ we define

$$\begin{aligned}\mathcal{X}^\eta_{\tilde{L}} &:= C_{\eta,\tilde{L}} \times C_{-\eta,\tilde{L}}\\ ||W||^\eta_{\tilde{L}} &:= ||w_1||_{\eta,\tilde{L}} + ||w_2||_{-\eta,\tilde{L}}\end{aligned} \tag{3.10}$$

We will consider spaces for fixed $\tilde{L}$ and variational solutions with $L \le \tilde{L}$ to be able to conclude smoothness of a fixed point in L in a fixed space.

Notation *Throughout this section, we fix an arbitrary η so that $0 < \eta < \kappa$, and only consider the spaces $\mathcal{X}^\eta_L$ as well as $\mathcal{X}^0_L$. For convenience we set $\mathcal{X}_L := \mathcal{X}^\eta_L$ and $||\cdot||_L := ||\cdot||^\eta_L$. In addition, we omit dependence of constants on η.*

The norms on $\mathcal{X}^0_L$ and $\mathcal{X}_L$ are equivalent for $L < \infty$, but one of the constants relating the norms diverges continuously as $L \to \infty$. Hence, for $L < \infty$ the spaces coincide as sets and smoothness in one space implies it in the other. For any $0 < L_1 \le L_2$ we have

$$\begin{aligned}\sup\nolimits_{0\le\zeta\le L_1} |w(\xi)| &\le ||w||_{\eta,L_1} \le ||w||_{\eta,L_2}\\ \sup\nolimits_{-L_1\le\zeta\le 0} |w(\xi)| &\le ||w||_{-\eta,L_1} \le ||w||_{-\eta,L_2},\end{aligned} \tag{3.11}$$

hence $||W||^0_{L_1} \le ||W||_{L_1}$ and for $W \in \mathcal{X}_{L_1} \cap \mathcal{X}_{L_2}$ we have $||W||_{L_1} \le ||W||_{L_2}$. Analogously, we obtain $||W||_L \le \mathrm{e}^{\eta L}||W||^0_L$ for any L.

Notation *We denote by $B_\rho(y)$ be the open ball of radius ρ centered at y in a metric space given by the context. For the spaces introduced above, we denote $B_\rho(W;L) := B_\rho(W) \subset \mathcal{X}_L$, and $B^0_\rho(W;L) := B_\rho(W) \subset \mathcal{X}^0_L$.* So for $L_1 \le L_2$ and $W \in \mathcal{X}_{L_2}$ it holds that as sets $B_\rho(W;L_1) \supset B_\rho(W;L_2)$, and for any $L > 0$ we have $B^0_\rho(W;L) \supset B_\rho(W;L) \supset B^0_{\rho\exp(-\eta L)}(W;L)$.

To obtain smoothness in L, we will fix the space, i.e. an $\tilde{L} \in I_b$. By the estimate $||W||_L \le ||W||_{\tilde{L}}$ for any $L < \tilde{L}$ uniform estimates in $\tilde{L}$ that only depend on $\tilde{L} - L$ are possible. Given $\tilde{L} \in I_b$, let $\tilde{I}$ be the interval in I_b containing $\tilde{L}$,

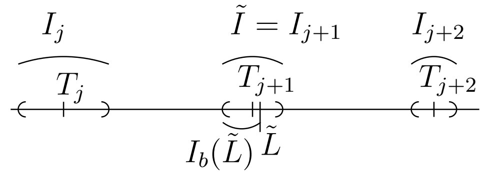

Fig. 7. Image of the definitions for sets of approximate semi travel times.

and denote $I_b(\tilde{L}) := \tilde{I} \cap (0, \tilde{L})$. See figure 7 for these definitions. For L_0 from lemma 2, denote $I_b^0 := I_b \cap (L_0, \infty)$, $I_b^0(\tilde{L}) := I_b(\tilde{L}) \cap I_b^0$. The set I_b^0 can be thought of as the set of almost phase coherent approximate semi-travel times L between sections Σ_1 and Σ_2. In the course of this section more restrictions on L will be imposed and the lower bound will be increased.

The precise statement of theorem 1 needs some more notation. We define

$$\Lambda_{\tilde{L}}^{\delta} := B_{\delta \exp(-\eta \tilde{L})}(0) \cap \Lambda \subset \mathbb{R}^d.$$

The parts of the initial conditions that decay towards γ in light of the trichotomy estimates (2.7) will be taken as parameters and denoted by

$$W_0 := \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} := \begin{pmatrix} w_{+1}^{\mathrm{s}}(0) \\ w_{-2}^{\mathrm{u}}(0) \end{pmatrix}.$$

To compensate not using transverse sections Σ_j for the initial conditions of the variations, see remark 3, we consider shifted variations: given a solution u to (2.1) define

$$\begin{aligned} w_1(\xi;\sigma) &:= u(\xi+\sigma) - q_1(\xi) \\ w_2(\xi;\sigma,L) &:= u(2L+\sigma+\xi) - q_2(\xi) \\ W(\sigma,L) &:= (w_1(\cdot;\sigma), w_2(\cdot;\sigma,L)) \\ W_0(\sigma,L) &:= (P_{+1}^{\mathrm{s}}(0)w_1(0;\sigma), P_{-2}^{\mathrm{u}}(0)w_2(0;\sigma,L)) \end{aligned} \tag{3.12}$$

The solution operator $\mathcal{G}$ referred to in the following theorem will be defined in (3.14) below.

Theorem 1 *Assume hypothesis 1. There exist positive constants $\epsilon, L_1, \delta, C > 0$ such that the following holds for all $\tilde{L} \in I_b^0 \cap (L_1, \infty)$, $L \in I_b^0(\tilde{L}) \cap (L_1, \infty)$ and $\mu \in \Lambda_{\tilde{L}}^{\delta}$, $W_0 \in B_\delta(0) \subset E_{+1}^{\mathrm{s}}(0) \times E_{-2}^{\mathrm{u}}(0)$. There exists $W^1(W_0, \mu, L) \in \mathcal{X}_{\tilde{L}}^0$, which is C^k in (W_0, μ, L) and a fixed point of $\mathcal{G}$. Any fixed point of $\mathcal{G}$ for $|\mu| < \delta$, $|W_0| < \delta$ and $L \geq L_1$ satisfies*

$$||W^1(W_0,\mu,L)||_{\tilde{L}} \leq C(|\mu| + |W_0| + \mathrm{e}^{(\eta-\kappa)\tilde{L}}). \tag{3.13}$$

Let u be a solution to (2.1) for $\mu \in \Lambda$ and assume there are $\sigma \in \mathbb{R}$ and

$L \geq L_1$ such that $\mu \in \Lambda_L^\delta$ and for the variations $||W(\sigma, L)||_L^0 < \epsilon$. There exists $\ell_L \in I_b^0 \cap (L_1, \infty)$ and a unique $\sigma_L = \sigma + O(|w_{+1}^{\mathrm{c}}(L;\sigma)|)$ which is C^k in L, so that $W(\sigma_L, \ell_L) = W^1(W_0(\sigma_L, \ell_L), \mu, \ell_L)$.

With L_1 from the theorem, we define $I_b^1 := I_b^0 \cap (L_1, \infty)$ and $I_b^1(\tilde{L}) := I_b^0(\tilde{L}) \cap I_b^1$.

The proof will be given at the end of this section. Notice that for fixed μ, the parameters W_0 and L give n dimensions. Since the ambient space is n-dimensional, a one-dimensional set of solutions given by the theorem is necessarily related by time shifts. Counting parameters and boundary/initial conditions for q_1 and q_2 there are $2n + 2$ degrees of freedom. We aim to satisfy (3.9) using n initial conditions, leaving the rest for the matching steps 2a-2c.

Remark 4 *It follows from (3.13) that for $\mu, W_0 \to 0$ and $L \to \infty$ fixed points of $\mathcal{G}$ converge to the set of heteroclinic points $\{q_1(\xi)|\xi \geq 0\} \cup \{q_2(\xi)|\xi \leq 0\}$. While solutions to (2.1) with approximate semi travel time $L \notin B_\epsilon(I_b^1)$ are beyond the reach of theorem 1, such solutions cannot lie arbitrarily close to q_1 and q_2 simultaneously.*

For an approach to the uniqueness of small glued solutions using an unambiguous semi travel time L we refer to remark 3. Notice that the ambiguity in defining the approximate semi travel time is resolved in theorem 1 by the unique small time shift σ_L, and parameters W_0, L, μ in ranges given in the theorem identify glued solutions with small variation uniquely up to this time shift.

The formulation of (3.9) as a contraction fixed point problem should be set up so we can expect small solutions. The following definition of a solution operator for glued solutions will be justified in lemma 4. The solution operator $\mathcal{G}$, on the spaces $\mathcal{X}_{\tilde{L}}$, cf. (3.10), will be of the form

$$\mathcal{G}(W; W_0, \mu, L) = \mathcal{A}(L)W_0 + \mathcal{N}(W; \mu, L) + \mathcal{B}(L). \tag{3.14}$$

Here $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{B}$ are functions of ξ_+ and ξ_- as in (3.7). We denote the evolutions Φ_1 and Φ_2 and nonlinearities of (2.5) projected with the trichotomies by

$$\Phi_{+1}^{\mathrm{s/c/u}}(\xi, \zeta) := P_{+1}^{\mathrm{s}}(\xi)\Phi_1(\xi, \zeta)$$
$$\Phi_{-2}^{\mathrm{s/c/u}}(\xi, \zeta) := P_{-2}^{\mathrm{s}}(\xi)\Phi_2(\xi, \zeta)$$

We assume $w_{+1}^{\mathrm{c}}(L) = 0$ and define, or rather conclude from (3.9) and the variation of constants formula, the three parts of the solution operator to respect the boundary condition (3.9). Firstly, define the linear term

$$\mathfrak{c}_1(W_0, L) := \Phi_{-2}^{\mathrm{u}}(-L, 0)\tilde{w}_2 - \Phi_{+1}^{\mathrm{s}}(L, 0)\tilde{w}_1 \ , \tag{3.15}$$

$$\mathcal{A}(L)\binom{\xi_+}{\xi_-}W_0 := \begin{pmatrix} \Phi^{\rm s}_{+1}(\xi_+,0)\tilde{w}_1 + \Phi^{\rm u}_{+1}(\xi_+,L)P^{\rm u}_L\mathfrak{c}_1(W_0,L) \\ \Phi^{\rm u}_{-2}(\xi_-,0)\tilde{w}_2 + \Phi^{\rm sc}_{-2}(\xi_-,-L)P^{\rm sc}_L(-\mathfrak{c}_1(W_0,L)) \end{pmatrix}. \quad (3.16)$$

Next we define a nonlinear coupling term $\mathfrak{c}_2 = \mathfrak{c}_2(W;\mu,L)$ and the nonlinearity $\mathcal{N} = \mathcal{N}(W;\mu,L)$.

$$\mathfrak{c}_2 := \int_0^{-L} \Phi^{\rm u}_{-2}(-L,\zeta)g_2(w_2(\zeta),\zeta;\mu)\mathrm{d}\zeta - \int_0^{L} \Phi^{\rm s}_{+1}(L,\zeta)g_1(w_1(\zeta),\zeta;\mu)\mathrm{d}\zeta, \quad (3.17)$$

$$\mathcal{N}\binom{\xi_+}{\xi_-} := \left(\begin{array}{c} \int_0^{\xi_+} \Phi^{\rm s}_{+1}(\xi_+,\zeta)g_1(w_1(\zeta),\zeta;\mu)\mathrm{d}\zeta \\ + \int_L^{\xi_+} \Phi^{\rm cu}_{+1}(\xi_+,\zeta)g_1(w_1(\zeta),\zeta;\mu)\mathrm{d}\zeta + \Phi^{\rm u}_{+1}(\xi_+,L)P^{\rm u}_L\mathfrak{c}_2 \\ \hline \int_{-L}^{\xi_-} \Phi^{\rm sc}_{-2}(\xi_-,\zeta)g_2(w_2(\zeta),\zeta;\mu)\mathrm{d}\zeta - \Phi^{\rm sc}_{-2}(\xi_-,-L)P^{\rm sc}_L\mathfrak{c}_2 \\ + \int_0^{\xi_-} \Phi^{\rm u}_{-2}(\xi_-,\zeta)g_2(w_2(\zeta),\zeta;\mu)\mathrm{d}\zeta \end{array}\right). \quad (3.18)$$

Notice that $\mathcal{N}(0;0,L) \equiv 0$, because $g_j(0,\cdot;0) \equiv 0$ for $j = 1,2$. Finally, the boundary term of the solution operator, which captures the rest of the gluing condition (3.9), is given by

$$\mathcal{B}(L)\binom{\xi_+}{\xi_-} := \begin{pmatrix} \Phi^{\rm u}_{+1}(\xi_+,L)P^{\rm u}_L b_L \\ -\Phi^{\rm sc}_{-2}(\xi_-,-L)P^{\rm sc}_L b_L \end{pmatrix}. \quad (3.19)$$

By definition $\mathcal{G}(\cdot;W_0,\mu,L)$ maps $\mathcal{X}_L$ into itself for any W_0, μ and $L < \infty$.

The following lemma establishes the connection of fixed points of $\mathcal{G}$ with glued solutions.

Lemma 4 *Assume hypothesis 1 and take L_0, ϵ_0 as in lemma 2. Let W be a fixed point of the operator $\mathcal{G}(\cdot;W_0,\mu,L)$, as defined in (3.14) for some $W_0 \in E^{\rm s}_{+1}(0) \times E^{\rm u}_{-2}(0)$, $\mu \in \Lambda$ and $L \geq L_0$ such that $|L - T_j| \leq \epsilon_0$ for some j. Then W solves (2.5) and (3.9).*

There exists $\rho_0 > 0$, such that the following holds. Let u be a solution to (2.1) and assume there are σ_0 and $L \geq L_0$ so that $|L - T_j| < \epsilon_0$ for some j and $|w_1(L;\sigma_0)| < \rho_0$. There exists a locally unique C^k function $\sigma_L = O(w^{\rm c}_{+1}(L;\sigma_0))$ which solves $w^{\rm c}_{+1}(L;\sigma) = 0$ so that $W(\sigma_L, L) = \mathcal{G}(W(\sigma_L,L);W_0(\sigma_L,L),\mu,L)$.

Proof. Let W be a fixed point for some W_0, μ, L, i.e. at $\xi_\pm = \pm L$ we have

$$W(L,-L) = \mathcal{A}(L)(L,-L)W_0 + \mathcal{N}(W;\mu,L)(L,-L) + \mathcal{B}(L)(L,-L).$$

To check the gluing (boundary) condition (3.9), we need to compute the difference of the first and second component. The left hand side yields $w_1(L) - w_2(-L)$, hence the right hand side's difference should be b_L. Using lemma 2 it holds that $P^{\mathrm{u}}_{+1}P^{\mathrm{u}}_L = P^{\mathrm{u}}_L$ and $P^{\mathrm{s}}_{-2}P^{\mathrm{s}}_L = P^{\mathrm{s}}_L$ and $P^{\mathrm{c}}_{-2}P^{\mathrm{c}}_L = P^{\mathrm{c}}_L$. So the parts of $\mathcal{G}$ at $\xi_\pm = \pm L$ are

$$\begin{aligned}
\mathcal{A}(L)(L,-L)W_0 &= \begin{pmatrix} \Phi^{\mathrm{s}}_{+1}(L,0)\tilde{w}_1 + P^{\mathrm{u}}_L \mathfrak{c}_1(W_0,L) \\ -P^{\mathrm{sc}}_L \mathfrak{c}_1(W_0,L) + \Phi^{\mathrm{u}}_{-2}(-L,0)\tilde{w}_2 \end{pmatrix} \\
\mathcal{N}(W;\mu,L)(L,-L) &= \\
&= \begin{pmatrix} \int\limits_0^L \Phi^{\mathrm{s}}_{+1}(L,\zeta)g_1(w_1(\zeta),\zeta;\mu)\mathrm{d}\zeta + P^{\mathrm{u}}_L \mathfrak{c}_2(W;L,\mu) \\ \hline -P^{\mathrm{sc}}_L \mathfrak{c}_2(W;L,\mu) + \int\limits_0^{-L} \Phi^{\mathrm{u}}_{-2}(-L,\zeta)g_2(w_2(\zeta),\zeta;\mu)\mathrm{d}\zeta \end{pmatrix} \\
\mathcal{B}(L)(L,-L) &= \begin{pmatrix} P^{\mathrm{u}}_L b_L \\ -P^{\mathrm{sc}}_L b_L \end{pmatrix}.
\end{aligned}$$

By (2.8), the component's difference in $\mathcal{B}$ yields b_L. We see from (2.8) and the definitions of $\mathfrak{c}_1(W_0,L)$ and $\mathfrak{c}_2(W;L,\mu)$ that the components' differences in $\mathcal{A}$ and $\mathcal{N}$ vanish. In particular, this shows that the image of $\mathcal{G}(\cdot, W_0, \mu, L)$ for any function satisfies (3.9). Finally, differentiation shows that a fixed point solves (2.5).

Conversely, assume $W(\sigma, L)$ is a glued solution pair with $w^{\mathrm{c}}_{+1}(L;\sigma) = 0$ and $L \geq L_0$, $|L - T_j| < \epsilon_0$ for some j and $\mu \in \Lambda$. Set $W_0 := (w^{\mathrm{s}}_{+1}(0;\sigma), w^{\mathrm{u}}_{-2}(0;\sigma,L))$, so by definition of $\mathcal{G}$ we have

$$\begin{aligned}
w^{\mathrm{s}}_{+1}(0;\sigma) &= P^{\mathrm{s}}_{+1}(0)\mathcal{G}_1(W(\sigma,L);W_0,\mu,L)(0) \\
w^{\mathrm{u}}_{-2}(0;\sigma,L) &= P^{\mathrm{u}}_{-2}(0)\mathcal{G}_2(W(\sigma,L);W_0,\mu,L)(0).
\end{aligned}$$

Since $W(\sigma, L)$ satisfies (3.9) it follows that $w_1(L;\sigma) - w_2(-L;\sigma,L) = b_L$. By lemma 2 this equation can be decomposed into

$$\begin{aligned}
w^{\mathrm{u}}_{+1}(L;\sigma,L) &= P^{\mathrm{u}}_L(b_L + w^{\mathrm{u}}_{-2}(-L;\sigma,L) - w^{\mathrm{s}}_{+1}(L;\sigma)) \\
w^{\mathrm{c}}_{-2}(-L;\sigma,L) &= P^{\mathrm{c}}_L(b_L + w^{\mathrm{s}}_{+1}(L;\sigma) - w^{\mathrm{u}}_{-2}(-L;\sigma,L)) \\
w^{\mathrm{s}}_{-2}(-L;\sigma,L) &= P^{\mathrm{s}}_L(b_L + w^{\mathrm{s}}_{+1}(L;\sigma) - w^{\mathrm{u}}_{-2}(-L;\sigma,L)).
\end{aligned}$$

At $\xi = L$ the variation of constants for $w_1(\xi;\sigma)$ starting at $w_1(0;\sigma)$ gives

$$w_1(L;\sigma) = \Phi(L,0)w_1(0;\sigma) + \int_0^L \Phi(L,\zeta)g_1(w_1(\zeta;\sigma),\zeta,\mu)\mathrm{d}\zeta.$$

Project the right hand side with P_L^{s}, P_L^{c}, P_L^{u} respectively and use the above decomposition of $w_{+1}^{\mathrm{u}}(L;\sigma)$ as well as the definition of the coupling terms $\mathfrak{c}_1$ and $\mathfrak{c}_2$. This yields precisely the terms of $\mathcal{G}_1(W(\sigma,L),W_0,\mu,L)(L)$. Similarly, we obtain $\mathcal{G}_2(W(\sigma,L),W_0,\mu,L)(-L)$. Hence, at $\xi_\pm=\pm L$ both $W(\sigma,L)$ and $\mathcal{G}(W(\sigma,L),W_0,\mu,L)$ solve the same initial value problem and hence coincide.

It remains to solve $w_{+1}^{\mathrm{c}}(L;\sigma)=0$ for σ near σ_0 given $|w_{+1}^{\mathrm{c}}(L;\sigma,L)|$ is sufficiently small. Note $\ker P_{+1}^{\mathrm{c}}(L)=E_{+1}^{\mathrm{s}}(L)\oplus E_{+1}^{\mathrm{u}}(L)$ is a linear $n-1$ dimensional subspace, transverse to the flow, because $\mathrm{Rg}(P_{+1}^{\mathrm{c}}(L))=\mathrm{span}\{\dot q_1(L)\}$. By continuity there exists $\rho_0>0$ and a constant $C>0$ such that for $|u(L+\sigma_0)-q_1(L)|<\rho_0$ we have $|\frac{\mathrm{d}}{\mathrm{d}\sigma}|_{\sigma=\sigma_0}u(L+\sigma)|\geq C\min\{\dot q_1(\xi)\,|\,0\leq\xi<T_\gamma\}>0$ uniformly in L. The implicit function theorem applies and $u(L+\sigma_L)\in q_1(L)+\ker P_{+1}^{\mathrm{c}}(L)$ for a unique σ_L near σ_0. By smoothness of the flow σ_L is $O(w_{+1}^{\mathrm{c}}(L;0))$ and C^k in L. Since $w_1(L;\sigma_0)=u(L+\sigma_0)-q_1(L)$ the claim follows. $\square$

For the local existence and uniqueness up to time shifts of glued solutions, this lemma allows to assume $w_{+1}^{\mathrm{c}}(L)=0$ without loss of generality, so we focus on the following fixed point equation, which we want to solve by means of proposition 1.

$$W=\mathcal{G}(W;W_0,\mu,L) \tag{3.20}$$

We aim to estimate some norm of $\mathcal{G}$ uniformly in $L\in I_b^0(\tilde L)$ for all $\tilde L\in I_b^0$. However, the trichotomy estimates (2.7) do not provide uniform control of the center direction, which is integrated in $\mathcal{N}$. Therefore, we make use of the weighted spaces $\mathcal{X}_L$ introduced in (3.10).

Since $\mathcal{G}(W;W_0,\mu,L)-\mathcal{G}(V;W_0,\mu,L)=\mathcal{N}(W;\mu,L)-\mathcal{N}(V;\mu,L)$, differentiability in W and contraction properties of $\mathcal{G}$ depend only on $\mathcal{N}$, and we examine these first.

Lemma 5 *Assume hypothesis 1 and take L_0, ϵ_0 as in lemma 2. For any $\tilde L\geq L\geq L_0$, $|L-T_j|<\epsilon_0$ for some j, the nonlinearity $\mathcal{N}(\cdot;\mu,L):\mathcal{X}_{\tilde L}\to\mathcal{X}_{\tilde L}$ is C^k in W, μ and L, and there exists a constant C such that*

$$\begin{aligned}
||\partial_W\mathcal{N}(W;\mu,L)||_{\tilde L}&\leq C\mathrm{e}^{(\kappa+\eta)(\tilde L-L)}(||W||_{\tilde L}+\mathrm{e}^{\eta L}|\mu|)\\
||\mathcal{N}(W;\mu,L)||_{\tilde L}&\leq C\mathrm{e}^{(\kappa+\eta)(\tilde L-L)}\Big(||W||_{\tilde L}||W||_{\tilde L}^0+|\mu|\,||W||_{\tilde L}+|\mu|\Big).
\end{aligned}$$

Proof. We first find a family of bounded linear operators $\mathcal{I}(L):\mathcal{X}_{\tilde L}\to\mathcal{X}_{\tilde L}$, C^k for any $\tilde L$, L as in the statement, and a family of functions $G(\cdot;\mu):\mathcal{X}_{\tilde L}\to\mathcal{X}_{\tilde L}$ of class C^k in W and μ such that $\mathcal{N}(W;\mu,L)=\mathcal{I}(L)G(W;\mu)$. This implies the claimed smoothness of $\mathcal{N}$, and the estimates will follow. Definition (3.18) implies that given any function

$$g = (g_1, g_2) : \mathcal{X}_{\tilde{L}} \to \mathcal{X}_{\tilde{L}},\ g_1 : C_{\eta,\tilde{L}} \to C_{\eta,\tilde{L}},\ g_2 : C_{-\eta,\tilde{L}} \to C_{-\eta,\tilde{L}}$$

we can define the desired linear operator $\mathcal{I}(L)$ to be

$$\mathcal{I}(L)g\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} := \left(\begin{array}{l} \int_0^{\xi_+} \Phi^{\mathrm{s}}_{+1}(\xi_+,\zeta)g_1(\zeta)\mathrm{d}\zeta \\ + \int_L^{\xi_+} \Phi^{\mathrm{u}}_{+1}(\xi_+,\zeta)g_1(\zeta)\mathrm{d}\zeta + \Phi^{\mathrm{u}}_{+1}(\xi_+,L)P^{\mathrm{u}}_L\mathfrak{c}(g;L) \\ + \int_L^{\xi_+} \Phi^{\mathrm{c}}_{+1}(\xi_+,\zeta)g_1(\zeta)\mathrm{d}\zeta \\ \hline \int_{-L}^{\xi_-} \Phi^{\mathrm{s}}_{-2}(\xi_-,\zeta)g_2(\zeta)\mathrm{d}\zeta - \Phi^{\mathrm{s}}_{-2}(\xi_-,-L)P^{\mathrm{s}}_L\mathfrak{c}(g;L) \\ + \int_0^{\xi_-} \Phi^{\mathrm{u}}_{-2}(\xi_-,\zeta)g_2(\zeta)\mathrm{d}\zeta \\ + \int_{-L}^{\xi_-} \Phi^{\mathrm{c}}_{-2}(\xi_-,\zeta)g_2(\zeta)\mathrm{d}\zeta - \Phi^{\mathrm{c}}_{-2}(\xi_-,-L)P^{\mathrm{c}}_L\mathfrak{c}(g;L) \end{array}\right)$$

$$\text{where } \mathfrak{c}(g;L) := \int_0^{-L} \Phi^{\mathrm{u}}_{-2}(-L,\zeta)g_2(\zeta)\mathrm{d}\zeta - \int_0^{L} \Phi^{\mathrm{s}}_{+1}(L,\zeta)g_1(\zeta)\mathrm{d}\zeta.$$

In the following estimates we frequently use (2.7) and that for $0 \le \zeta \le L \le \tilde{L}$ we have the pointwise estimate

$$|g_1(\zeta)| = \mathrm{e}^{-\eta\zeta}\mathrm{e}^{\eta\zeta}|g_1(\zeta)| \le \mathrm{e}^{-\eta\zeta} \sup_{0\le\zeta\le L} \mathrm{e}^{\eta\zeta}|g_1(\zeta)| = \mathrm{e}^{-\eta\zeta}||g_1||_{\eta,\tilde{L}},$$

similarly $|g_2(\zeta)| \le \mathrm{e}^{\eta\zeta}||g_2||_{-\eta,\tilde{L}}$ for $-\tilde{L} \le -L \le \zeta \le 0$. As to the coupling term, we have

$$\begin{aligned} |\mathfrak{c}(g;L)| &= |\int_0^{-L} \Phi^{\mathrm{u}}_{-2}(-L,\zeta)g_2(\zeta)\mathrm{d}\zeta - \int_0^{L} \Phi^{\mathrm{s}}_{+1}(L,\zeta)g_1(\zeta)\mathrm{d}\zeta| \\ &\le K_2 \int_{-L}^{0} \mathrm{e}^{-\kappa(L+\zeta)}|g_2(\zeta)|\mathrm{d}\zeta + K_2 \int_0^L \mathrm{e}^{-\kappa(L-\zeta)}|g_2(\zeta)|\mathrm{d}\zeta \\ &\le K_2 \int_{-L}^{0} \mathrm{e}^{-\kappa(L+\zeta)}\mathrm{e}^{\eta\zeta}||g_2||_{-\eta,\tilde{L}}\mathrm{d}\zeta + K_2 \int_0^L \mathrm{e}^{-\kappa(L-\zeta)}\mathrm{e}^{-\eta\zeta}||g_1||_{\eta,\tilde{L}}\mathrm{d}\zeta \\ &\le K_2 \left(\frac{\mathrm{e}^{-\eta L} - \mathrm{e}^{-\kappa L}}{\kappa - \eta}||g_2||_{-\eta,\tilde{L}} + \frac{\mathrm{e}^{-\eta L} - \mathrm{e}^{-\kappa L}}{\kappa - \eta}||g_1||_{\eta,\tilde{L}}\right) \\ &\le \frac{K_2}{\kappa - \eta}\mathrm{e}^{-\eta L}||g||_{\tilde{L}}. \end{aligned} \tag{3.21}$$

Hence the coupling is exponentially small, and applying the projected evolutions we obtain

$$\sup_{0\le\xi\le\tilde{L}} |\Phi^{\mathrm{u}}_{+1}(\xi,L)P^{\mathrm{u}}_L\mathfrak{c}(g;L)| \le \sup_{0\le\xi\le\tilde{L}} K_2\mathrm{e}^{-\kappa(L-\xi)}K_3\frac{K_2}{\kappa-\eta}\mathrm{e}^{-\eta L}||g||_{\tilde{L}}$$
$$\le C\mathrm{e}^{\kappa(\tilde{L}-L)}\mathrm{e}^{-\eta L}||g||_{\tilde{L}}, \tag{3.22}$$
$$\sup_{0\le\xi\le\tilde{L}} |\Phi^{\mathrm{sc}}_{-2}(\xi,L)P^{\mathrm{sc}}_L\mathfrak{c}(g;L)| \le C\mathrm{e}^{-\eta L}||g||_{\tilde{L}}. \tag{3.23}$$

Now we consider the six parts of $\mathcal{I}(L)$ without the projected and evolved coupling term aiming at uniform estimates in $\tilde{L}$, L. Note that the constants C may depend on η.

i)
$$\sup_{0\le\xi\le\tilde{L}} |\mathrm{e}^{\eta\xi}\int_0^\xi \Phi^{\mathrm{s}}_{+1}(\xi,\zeta)g_1(\zeta)\mathrm{d}\zeta| \le \sup_{0\le\xi\le\tilde{L}} \mathrm{e}^{\eta\xi}\int_0^\xi K_2\mathrm{e}^{-\kappa(\xi-\zeta)}|g_1(\zeta)|\mathrm{d}\zeta$$
$$\le K_2 \sup_{0\le\xi\le\tilde{L}} \mathrm{e}^{\eta\xi}\int_0^\xi \mathrm{e}^{-\kappa(\xi-\zeta)}\mathrm{e}^{-\eta\zeta}||g_1||_{\eta,\tilde{L}}\mathrm{d}\zeta$$
$$= K_2 \sup_{0\le\xi\le\tilde{L}} \mathrm{e}^{\eta\xi}\frac{\mathrm{e}^{-\eta\xi}-\mathrm{e}^{-\kappa\xi}}{\kappa-\eta}||g_1||_{\eta,\tilde{L}} \le \frac{K_2}{\kappa-\eta}||g_1||_{\eta,\tilde{L}}$$

ii)
$$\sup_{0\le\xi\le\tilde{L}} |\mathrm{e}^{\eta\xi}\int_L^\xi \Phi^{\mathrm{u}}_{+1}(\xi,\zeta)g_1(\zeta)\mathrm{d}\zeta| \le \sup_{0\le\xi\le\tilde{L}} \mathrm{e}^{\eta\xi}|\int_L^\xi K_2\mathrm{e}^{-\kappa(\zeta-\xi)}|g_1(\zeta)|\mathrm{d}\zeta|$$
$$\le K_2 \sup_{0\le\xi\le\tilde{L}} \mathrm{e}^{\eta\xi}|\int_L^\xi \mathrm{e}^{-\kappa(\zeta-\xi)}\mathrm{e}^{-\eta\zeta}||g_1||_{\eta,\tilde{L}}\mathrm{d}\zeta|$$
$$= K_2 \sup_{0\le\xi\le\tilde{L}} \mathrm{e}^{\eta\xi}\frac{|\mathrm{e}^{-\kappa(L-\xi)-\eta L}-\mathrm{e}^{-\eta\xi}|}{\kappa+\eta}||g_1||_{\eta,\tilde{L}}$$
$$\le C\mathrm{e}^{(\kappa+\eta)(\tilde{L}-L)}||g_1||_{\eta,\tilde{L}}$$

iii) The following shows the relevance of weighted spaces for uniform estimates:

$$\begin{aligned} &\sup\nolimits_{0\le\xi\le\tilde{L}} |\mathrm{e}^{\eta\xi}\textstyle\int_L^\xi \Phi^{\mathrm{c}}_{+1}(\xi,\zeta)g_1(\zeta)\mathrm{d}\zeta| \le \sup\nolimits_{0\le\xi\le\tilde{L}} K_2\mathrm{e}^{\eta\xi}|\textstyle\int_L^\xi \mathrm{e}^{-\eta\zeta}||g_1||_{\eta,\tilde{L}}\mathrm{d}\zeta| \\ = \;& K_2 \sup\nolimits_{0\le\xi\le\tilde{L}} \mathrm{e}^{\eta\xi}\tfrac{|\mathrm{e}^{-\eta\xi}-\mathrm{e}^{-\eta L}|}{\eta}||g_1||_{\eta,\tilde{L}} \le C\mathrm{e}^{\eta(\tilde{L}-L)}||g_1||_{\eta,\tilde{L}} \end{aligned}$$

iv)
$$\sup_{-\tilde{L}\le\xi\le 0} |\mathrm{e}^{-\eta\xi}\int_{-L}^{\xi} \Phi^{\mathrm{s}}_{-2}(\xi,\zeta)g_2(\zeta)\mathrm{d}\zeta| \le \sup_{-\tilde{L}\le\xi\le 0} \mathrm{e}^{-\eta\xi}|\int_{-L}^{\xi} K_2\mathrm{e}^{\kappa(\zeta-\xi)}\mathrm{e}^{\eta\zeta}||g_2||_{-\eta,\tilde{L}}\mathrm{d}\zeta|$$
$$= K_2 \sup_{-\tilde{L}\le\xi\le 0} \mathrm{e}^{-\eta\xi}\frac{|\mathrm{e}^{-\kappa(L+\xi)-\eta L}-\mathrm{e}^{\eta\xi}|}{\kappa+\eta}||g_2||_{-\eta,\tilde{L}} \le C\mathrm{e}^{(\kappa+\eta)(\tilde{L}-L)}||g_2||_{-\eta,\tilde{L}}$$

v)

$$\sup_{-\tilde{L}\le\xi\le 0} |\mathrm{e}^{-\eta\xi}\int\limits_0^{\xi} \Phi^{\mathrm{u}}_{-2}(\xi,\zeta)g_2(\zeta)\mathrm{d}\zeta| \le \sup_{-\tilde{L}\le\xi\le 0} K_2\mathrm{e}^{-\eta\xi}\int\limits_0^{\xi} \mathrm{e}^{\kappa(\xi-\zeta)}\mathrm{e}^{\eta\zeta}||g_2||_{-\eta,\tilde{L}}\mathrm{d}\zeta$$
$$= K_2 \sup_{-\tilde{L}\le\xi\le 0} \mathrm{e}^{-\eta\xi}\frac{\mathrm{e}^{\kappa\xi}-\mathrm{e}^{\eta\xi}}{\kappa-\eta}||g_2||_{-\eta,\tilde{L}} \le C||g_2||_{-\eta,\tilde{L}}$$

vi) Again, the center direction's estimate relies on the exponential weights:

$$\sup_{-\tilde{L}\le\xi\le 0} |\mathrm{e}^{-\eta\xi}\int_{-L}^{\xi} \Phi^{\mathrm{c}}_{-2}(\xi,\zeta)g_2(\zeta)\mathrm{d}\zeta| \le K_2 \sup_{-\tilde{L}\le\xi\le 0} \mathrm{e}^{-\eta\xi}|\int_{-L}^{\xi} \mathrm{e}^{\eta\zeta}||g_2||_{-\eta,\tilde{L}}\mathrm{d}\zeta|$$
$$\le K_2 \sup_{-\tilde{L}\le\xi\le 0} \frac{\mathrm{e}^{-\eta\xi}}{\eta}|\mathrm{e}^{\eta\xi}-\mathrm{e}^{-\eta L}|\,||g_2||_{-\eta,\tilde{L}} \le C\mathrm{e}^{\eta(\tilde{L}-L)}||g_2||_{-\eta,\tilde{L}}$$

By i) to vi) and (3.23), (3.22) $\mathcal{I}(L)$ is continuous in $\mathcal{X}_{\tilde{L}}$ for $\tilde{L}\ge L\ge L_0$ with

$$||\mathcal{I}(L)||_{\mathcal{L}(\mathcal{X}_{\tilde{L}},\mathcal{X}_{\tilde{L}})} \le C_{\mathcal{I}}\mathrm{e}^{\kappa(\tilde{L}-L)}. \tag{3.24}$$

Notice that $\xi\mapsto P^{\mathrm{s/c/u}}_{+1/-2}(\xi)v$ is C^k for any v, cf. proof of lemma 1.1 in [30]. Together with the smoothness of g_j, $j=1,2$ we conclude that $\mathcal{I}(L)$ is C^k in L. As to the specific Nemitskij operator for $\mathcal{N}(W;\mu,L)$, i.e.

$$G(W;\mu)(\xi_+,\xi_-) = (\, g_1(w_1(\xi_+),\xi_+;\mu)\ ,\ g_2(w_2(\xi_-),\xi_-;\mu)\,),$$

it follows e.g. from [30], Lemma 3.1, that $G(W;\mu)$ is C^k in W and μ on $\mathcal{X}_{\tilde{L}}$. Together, we conclude $\mathcal{N}(W;\mu,L)=\mathcal{I}(L)G(W;\mu,L)$ is C^k in W and μ.

The estimate for $\mathcal{N}$ follows from (3.24) and

$$
\begin{aligned}
||G(W;\mu)||_{\tilde L} &= ||g_1(w_1(\cdot),\cdot;\mu)||_{\eta,\tilde L} + ||g_2(w_2(\cdot),\cdot;\mu)||_{-\eta,\tilde L}\\
&\le \sup_{0\le\xi\le\tilde L} \mathrm{e}^{\eta\xi}|g_1(w_1(\cdot),\cdot;\mu)| + \sup_{-\tilde L\le\xi\le 0} \mathrm{e}^{-\eta\xi}|g_2(w_2(\cdot),\cdot;\mu)|\\
&\le C\left(\sup_{0\le\xi\le\tilde L} \mathrm{e}^{\eta\xi}(|w_1(\xi)|^2 + |\mu|(|w_1(\xi)| + |q_1(\xi)-\gamma(\xi)|))\right.\\
&\quad \left. + \sup_{-\tilde L\le\xi\le 0} \mathrm{e}^{-\eta\xi}(|w_2(\xi)|^2 + |\mu|(|w_2(\xi)| + |q_2(\xi)-\gamma(\xi)|))\right)\\
&\le C\left(\sup_{0\le\xi\le\tilde L} |w_1(\xi)|\,||w_1||_{\eta,\tilde L} + |\mu|(||w_1||_{\eta,\tilde L} + ||q_1-\gamma(\cdot)||_{\eta,\tilde L})\right.\\
&\quad \left. + \sup_{-\tilde L\le\xi\le 0} |w_2(\xi)|\,||w_2||_{-\eta,\tilde L} + |\mu|(||w_2||_{-\eta,\tilde L} + ||q_2-\gamma(\cdot)||_{-\eta,\tilde L})\right)\\
&\le C\left(||W||_{\tilde L}||W||^0_{\tilde L} + |\mu|(||W||_{\tilde L} + ||q_1-\gamma||_{\eta,\tilde L} + ||q_2-\gamma||_{-\eta,\tilde L})\right)\\
&\le C\left(||W||_{\tilde L}||W||^0_{\tilde L} + |\mu|\,||W||_{\tilde L} + |\mu|\right)\\
&\le C\left(||W||^2_{\tilde L} + |\mu|\,||W||_{\tilde L} + |\mu|\right). \qquad (3.25)
\end{aligned}
$$

Here we used (2.6), and the final constant C absorbed $K_5 := \lim_{\tilde L\to\infty}||q_1 - \gamma||_{\eta,\tilde L} + ||q_2-\gamma||_{-\eta,\tilde L}$ which is bounded by (2.3). Analogously, the estimate for $\partial_W\mathcal{N}$ follows from

$$
||\partial_W G(\cdot;\mu)||_{\tilde L} = ||\partial_{w_1} g_1(w_1(\cdot),\cdot;\mu)||_{\eta,\tilde L} + ||\partial_{w_2} g_2(w_2(\cdot),\cdot;\mu)||_{-\eta,\tilde L}
$$

and from (2.6) with possibly adjusted K_1 so that

$$
||\partial_{w_1} g_1(w_1,\cdot;\mu)||_{\eta} + ||\partial_{w_2} g_2(w_2,\cdot;\mu)||_{-\eta} \le K_1(||W||_{\tilde L} + \mathrm{e}^{\eta\tilde L}|\mu|).
$$

□

Proof of theorem 1. The theorem follows from the implicit function theorem proposition 1 as follows. For the estimates (3.1), (3.4) in proposition 1, we first consider $\mathcal{A}$ as defined in (3.16). We frequently use the trichotomy estimates (2.7) which immediately give

$$
|\mathfrak{c}_1(W_0,L)| = |\Phi^{\mathrm{u}}_{-2}(-L,0)\tilde w_2 - \Phi^{\mathrm{s}}_{+1}(L,0)\tilde w_1| \le K_2\mathrm{e}^{-\kappa L}|W_0|, \qquad (3.26)
$$

and with $|P_L^{\mathrm{s/c/u}}| \le K_3$ from lemma 2 we can estimate

$$
\sup_{0\le\xi\le\tilde L} |\mathrm{e}^{\eta\xi}\Phi^{\mathrm{s}}_{+1}(\xi,0)\tilde w_1| \le \sup_{0\le\xi\le\tilde L} \mathrm{e}^{(\eta-\kappa)\xi}K_2|\tilde w_1| = K_2|\tilde w_1|
$$

$$\sup_{0\le\xi\le\tilde L} |\mathrm{e}^{\eta\xi}\Phi^{\mathrm{u}}_{+1}(\xi,L)P^{\mathrm{u}}_L\mathfrak{c}_1(W_0,L)| \le \sup_{0\le\xi\le\tilde L} \mathrm{e}^{\eta\xi+\kappa(\xi-L)}K_2K_3|\mathfrak{c}_1(W_0,L)|$$
$$\le\ C\mathrm{e}^{\eta\tilde L}\mathrm{e}^{\kappa(\tilde L-L)}|\mathfrak{c}_1(W_0,L)| \le C\mathrm{e}^{\kappa(\tilde L-L)}\mathrm{e}^{(\eta-\kappa)\tilde L}|W_0|$$

$$\sup_{-\tilde L\le\xi\le 0} |\mathrm{e}^{-\eta\xi}\Phi^{\mathrm{s}}_{-2}(\xi,-L)P^{\mathrm{s}}_L\mathfrak{c}_1(W_0,L)| \le \sup_{-\tilde L\le\xi\le 0} \mathrm{e}^{-\eta\xi-\kappa(\xi+L)}K_2K_3|\mathfrak{c}_1(W_0,L)|$$
$$=\ K_2K_3\mathrm{e}^{\kappa(\tilde L-L)}\mathrm{e}^{\eta\tilde L}|\mathfrak{c}_1(W_0,L)| \le C\mathrm{e}^{\kappa(\tilde L-L)}\mathrm{e}^{(\eta-\kappa)\tilde L}|W_0|$$
$$\sup_{-\tilde L\le\xi\le 0} |\mathrm{e}^{-\eta\xi}\Phi^{\mathrm{u}}_{-2}(\xi,0)\tilde w_2| \le\ \sup_{-\tilde L\le\xi\le 0} \mathrm{e}^{(-\eta+\kappa)\xi}K_2|\tilde w_2| = K_2|\tilde w_2|$$

$$\sup_{-\tilde L\le\xi\le 0} |\mathrm{e}^{\eta\xi}\Phi^{\mathrm{c}}_{-2}(\xi,-L)P^{\mathrm{c}}_L\mathfrak{c}_1(W_0,L)| \le \sup_{-\tilde L\le\xi\le 0} K_2K_3\mathrm{e}^{\eta\xi}|\mathfrak{c}_1(W_0,L)|$$
$$\le C\mathrm{e}^{(\eta-\kappa)\tilde L}|W_0|.$$

For $\tilde L \in I^0_b$ and $L \in I^0_b(\tilde L)$ we have $0 < \tilde L - L < \epsilon_0 < T_\gamma/2$, and with $|W_0| = |\tilde w_1| + |\tilde w_2|$ we obtain a constant C such that

$$||\mathcal{A}(L)W_0||_{\tilde L} \le C|W_0|. \tag{3.27}$$

Together with lemma 5 we obtain the estimates (3.1) needed in proposition 1. Now consider the constant boundary term $\mathcal{B}$ and its projections.

$$\sup_{0\le\xi\le\tilde L} |\mathrm{e}^{\eta\xi}\Phi^{\mathrm{u}}_{+1}(\xi,L)P^{\mathrm{u}}_L b_L| \le\ \sup_{0\le\xi\le\tilde L} \mathrm{e}^{-\kappa(L-\xi)+\eta\xi}K_2K_3|b_L| \le C\mathrm{e}^{\kappa(\tilde L-L)}\mathrm{e}^{\eta\tilde L}|b_L|$$
$$\sup_{-\tilde L\le\xi\le 0} |\mathrm{e}^{-\eta\xi}\Phi^{\mathrm{s}}_{-2}(\xi,-L)P^{\mathrm{s}}_L b_L| \le\ \sup_{-\tilde L\le\xi\le 0} \mathrm{e}^{-\eta\xi-\kappa(\xi+L)}C|b_L| \le C\mathrm{e}^{\kappa(\tilde L-L)}\mathrm{e}^{\eta\tilde L}|b_L|$$
$$\sup_{-\tilde L\le\xi\le 0} |\mathrm{e}^{-\eta\xi}\Phi^{\mathrm{c}}_{-2}(\xi,-L)P^{\mathrm{c}}_L b_L| \le\ \sup_{-\tilde L\le\xi\le 0} \mathrm{e}^{-\eta\xi}K_2K_3|b_L| \le C\mathrm{e}^{\eta\tilde L}|b_L|$$

For any $\tilde L \in I^0_b$ and $L \in I^0_b(\tilde L)$ we have $\tilde L - L \le T_\gamma/2$ and so lemma 3 implies

$$||\mathcal{B}(L)||_{\tilde L} \le C\mathrm{e}^{\eta\tilde L}|b_L| \le C\mathrm{e}^{(\eta-\kappa)\tilde L} \tag{3.28}$$

and in particular $\mathcal{B}(L) \to 0$ as $L \to \infty$.

Let C_2 be the sum of the constant C from lemma 5 and that of (3.27), and set $C_1 := 1$. Choose $\tilde L_1 \ge L_0$ so that $C_1|\mathcal{B}(L)| \le 1/(8C_1C_2)$. Fix any $\tilde L \in I^0_b \cap (\tilde L_1,\infty)$ and define $I := I^0_b(\tilde L)$, $X = Y := \mathcal{X}_{\tilde L}$ and $Z := \Lambda^1_{\tilde L} \times \mathbb{R}^n$ so $y = W$, $z = (\mu, W_0)$. Set $Q(z,L) := \mathrm{Id} : Y \to Y$, $R(y,z,L) := -\mathcal{A}(L)W_0 - \mathcal{N}(y,z,L)$ and $S(L) := \mathcal{B}(L)$.

Proposition 1 applies and provides constants ϵ_1, $\tilde\delta_1$ and C independent of $\tilde L$ and the desired fixed point $W^1(W_0,\mu,L)$ of $\mathcal{G}$. This is unique in $B_{\epsilon_1}(0;\tilde L)$ and

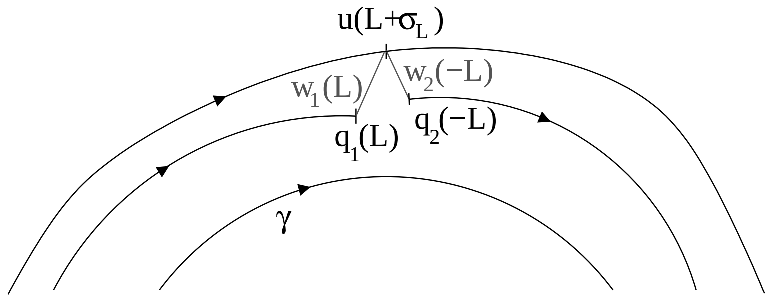

Fig. 8. Small glued solutions at the approximate semi-travel-time.

C^k in μ, W_0, L for $|W_0| < \delta_1/2$, $\tilde{L} \in I_b^0 \cap (\tilde{L}_1, \infty)$, $L \in I_b^0(\tilde{L}) \cap (\tilde{L}_1, \infty)$ and $|\mu| < \delta_1/2 \exp(-\eta\tilde{L})$. Note that while the parameter $|\mu|$ needs to be exponentially small to obtain a contraction, the estimate (3.3) holds for $|\mu| + |W_0| < \delta$ and $L \in I_b^0 \cap (\tilde{L}_1, \infty)$. Only the estimate for the derivative of G requires exponentially small μ, cf. lemma 5. Here, $|b_L| \leq C\mathrm{e}^{-\kappa\tilde{L}}$ and so (3.3) becomes

$$||W^1(W_0, \mu, L)||_{\tilde{L}} \leq C(|\mu| + |W_0| + \mathrm{e}^{(\eta-\kappa)\tilde{L}}).$$

Concluding the existence statement, take $\delta_1 = \tilde{\delta}_1/2$ and notice that lemma 4 implies that the fixed point is a glued solution.

As to the uniqueness, assume for the variation of a given solution $W(\sigma, L)$ and $\rho > 0$ that $W(\sigma, L) \in B_\rho^0(0; L)$. Recall the shift function σ_L from lemma 4 and that for $\rho \leq \rho_0$ it follows $W(\sigma_L, L) \in B_\rho^0(0; L)$ is a fixed point of $\mathcal{G}$. Firstly, we show that if ρ is sufficiently small and L is large enough, then we can find $\ell_L \in I_b^0$ such that $W(\sigma_{\ell_L}, \ell_L)$ is a fixed point of $\mathcal{G}$. Secondly, we prove that if a fixed point of $\mathcal{G}$ is small in $\mathcal{X}_L^0$ then it is small in $\mathcal{X}_L$ and hence the unique fixed point W^1.

For $W(\sigma, L) \in B_\rho^0(0; L)$ we can estimate (cf. figure 8)

$$\begin{aligned} |q_1(L) - q_2(-L)| &= |q_1(L) - u(L + \sigma) + u(L + \sigma) - q_2(-L)| \\ &= |w_2(-L; \sigma) - w_1(L; \sigma)| < \rho. \end{aligned}$$

Recall the definition of T_j and I_j from lemma 3. Assume $q_1(\xi_j) - q_2(-\xi_j) \to 0$ as $j \to \infty$ for some sequence of real numbers ξ_j. Since both have asymptotic phase zero at γ, this is equivalent to $\gamma(\xi_j) - \gamma(-\xi_j) \to 0$, which implies $\mathrm{dist}(\xi_j, \{T_i \,|\, i \geq 0\}) \to 0$. Hence, there is $C > 0$ and for all $\rho > 0$ there is a constant $\bar{L}_1 \geq \tilde{L}_1$ such that for all $L \geq \bar{L}_1$ there is a locally unique j_L with $|L - T_{j_L}| \leq C\rho$ (recall $W(\sigma, L) \in B_\rho^0(0; L)$). We next define ℓ_L: if $L \in I_b$ set $\ell_L := L$, otherwise $\ell_L := T_{j_L}$. It follows that $W(\sigma, \ell_L) \in B_{\rho'}^0(0; \ell_L)$, where $\rho' \leq C|L - T_{j_L}|$ accounts for shifting $u(\xi + 2L + \sigma)$ to $u(\xi + 2\ell_L + \sigma)$ relative to $q_2(\xi)$ for $\xi \in [-L, 0] \cup [-\ell_L, 0]$. Here C depends on $\sup\{|q_2(\xi)| \,|\, \xi \leq 0\}$, which is bounded because $q_2(\xi) - \gamma(\xi) \to 0$ as $\xi \to -\infty$. Thus, there are constants $C >$

0 and $\rho_1 > 0$ such that $\rho' \leq C\rho$ for all $0 < \rho \leq \rho_1$. Define $\rho_2 := \min\{\rho_1, \rho_0/C\}$ where ρ_0 is from lemma 4. If $\rho \leq \rho_2$ then $W(\sigma, \ell_L) \in B^0_{\rho_0}(0; \ell_L)$ and lemma 4 implies $W(\sigma_{\ell_L}, \ell_L)$ is a fixed point of $\mathcal{G}$. Since $\sigma_L = O(w^{\mathrm{c}}_{+1}(L, \sigma))$, we can choose ρ_2 so that in addition $W(\sigma_{\ell_L}, \ell_L) \in B^0_{\rho_1}(0; \ell_L)$.

We next show that a given fixed point which is small in $\mathcal{X}^0_L$ is actually small in $\mathcal{X}_L$ and conclude that $W(\sigma_{\ell_L}, \ell_L)$ is identical to one of the unique fixed points found above. Using lemma 5, (3.27) and (3.28) it follows for any fixed point $V(\sigma, L)$ of $\mathcal{G}$ with $L \in I^0_b \cap (\bar{L}_1, \infty)$ and $|W_0| + |\mu| < \delta_1$ that

$$\begin{aligned} ||V(\sigma, L)||_L \leq & C\Big(|V(\sigma, L)(0)| + ||V(\sigma, L)||_L(||V(\sigma, L)||^0_L + |\mu|) \\ & \quad + |\mu| + \mathrm{e}^{(\eta-\kappa)L}\Big) \\ \Rightarrow ||V(\sigma, L)||_L \leq & C\frac{\mathrm{e}^{(\eta-\kappa)L} + \rho_2 + |\mu|}{1 - C(\rho_2 + |\mu|)} =: c(L, \rho_2, \mu). \end{aligned}$$

By definition of $c(L, \rho, \mu)$ we can choose L_1, δ and ρ_3 so that $L_1 \geq \bar{L}_1$, $\rho_2 \geq \rho_3 > 0$ and such that $\delta_1 \geq \delta > 0$ and $c(L, \rho, \mu) \leq \epsilon_1$ for any $L \geq L_1$, $\rho \leq \rho_3$ and $|\mu| < \delta \exp(-\eta\tilde{L})$. Define $\epsilon := \min\{\rho_3, \epsilon_1, \delta_1\}$, so $W(\sigma, L) \in B^0_\epsilon(0; L)$ satisfies all conditions simultaneously and we obtain a fixed point $W(\sigma_{\ell_L}, \ell_L) \in B_{\epsilon_1}(0; \ell_L)$. By the contraction argument above, the fixed point of $\mathcal{G}(\cdot; W_0(\sigma_{\ell_L}, \ell_L), \mu, \ell_L)$ in this ball is unique, so $W(\sigma_{\ell_L}, \ell_L)$ is identical to $W^1(W_0(\sigma_{\ell_L}, \ell_L), \mu, \ell_L)$. Therefore this fixed point is actually unique in $B^0_\epsilon(0; \ell_L)$ and u can be written as in the theorem statement. □

3.2 Matching the glued solutions, homoclinic orbits

We want to find all W_0, μ and L such that the associated glued solution $W^1(W_0, \mu, L)$ yields a homoclinic orbit to p_0 by means of (3.8). For this, we now assume the di- and trichotomies from lemma 1 are maximally transverse at $\xi = 0$ and the heteroclinic cycle is codimension 2 or 1 as in definition 1.

Hypothesis 2 (codimension 2)

$$i_{p_0} = i_\gamma + 1, \quad \dim(E^{\mathrm{u}}_{-1}(0) \cap E^{\mathrm{s}}_{+1}(0)) = 1, \quad \dim(E^{\mathrm{s}}_{+2}(0) + E^{\mathrm{u}}_{-2}(0)) = n-2$$

Hypothesis 3 (codimension 1)

$$i_{p_0} = i_\gamma, \quad \dim(E^{\mathrm{u}}_{-1}(0) \cap E^{\mathrm{s}}_{+1}(0)) = 0, \quad \dim(E^{\mathrm{s}}_{+2}(0) + E^{\mathrm{u}}_{-2})(0) = n-1$$

We next assume that the parameters transversely unfold the intersection of stable and center-unstable manifolds: let a^j_0, $j = 1, 2$, be so that $E_2 := (E^{\mathrm{s}}_{+2} + E^{\mathrm{u}}_{-2}) = \mathrm{span}\{a^1_0, a^2_0\}^\perp$ and a^j, $j = 1, 2$, be solutions to the adjoint linear equation $\dot{a}^j = -(A_2(\xi))^* a^j$, with $a^j(0) = a^j_0$, $j = 1, 2$.

Hypothesis 4 *The following linear map* $\mathcal{M} : \mathbb{R}^d \to (E_2)^\perp$ *is invertible*

$$\mu \mapsto \sum_{j=1,2} \int_{-\infty}^{\infty} \langle f_\mu(q_2(\zeta);0)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta a_0^j.$$

Let $E_1 := E^{\mathrm{u}}_{-1}(0) \cap E^{\mathrm{s}}_{+1}(0)$ and $\tilde{Q} : E^{\mathrm{u}}_{-1}(0) \to E_1$ a projection with arbitrary kernel containing $E^{\mathrm{c}}_{+1}(0)$. Note that $E_1 = \{0\}$ and $\tilde{Q} = 0$ under hypothesis 3.

Theorem 2 *Assume hypotheses 2 or 3, and 1, 4. There exist positive constants* ϵ_4*,* δ_4*,* L_4 *and* C *such that for* $\tilde{L} \in I_b^1 \cap (L_4, \infty)$*,* $L \in I_b^1(\tilde{L}) \cap (L_4, \infty)$ *and* $\mathfrak{v} \in B_{\delta_4}(0) \subset E_1$ *the following holds. There exists countably infinite families of* C^k*-curves* $\mu(L, \mathfrak{v}) \in B_{\delta_4}(0) \subset \mathbb{R}^d$ *of parameters and* $h_{L,\mathfrak{v}} \in B_{\epsilon_4}(0; \tilde{L})$ *of 1–homoclinic orbits to* p_0 *in (2.1) such that*

$$\begin{aligned} |\mu(L, \mathfrak{v})| &\leq C(\mathrm{e}^{-2\kappa L} + |\mathfrak{v}|) \\ ||h_{L,\mathfrak{v}} - q_1||_{\eta,\tilde{L}} + ||h_{L,\mathfrak{v}}(2L + \cdot) - q_2||_{-\eta,\tilde{L}} &\leq C(\mathrm{e}^{(\eta-\kappa)\tilde{L}} + |\mathfrak{v}|). \end{aligned}$$

Given a 1–homoclinic solution h *to (2.1) for* $|\mu| < \delta_4$ *with variations* $W(\sigma; L) \in B^0_{\epsilon_4}(0; L)$*, there exist* $\ell_L = L + O(\mathrm{dist}(L, I_b^4))$ *and unique* $\sigma_L = O(w^{\mathrm{c}}_{+1}(L; \sigma))$ *which is* C^k *in* L*, such that* $h \equiv h_{\ell_L,\mathfrak{v}}(\cdot + \sigma_L)$ *and* $\mu = \mu(\ell_L, \mathfrak{v})$ *for* $\mathfrak{v} = \tilde{Q} w^{\mathrm{u}}_{-1}(0; \sigma_{\ell_L})$*.*

The more restrictive sets of almost phase coherent parameters are $I_b^4 := I_b^1 \cap (L_4, \infty)$ and $I_b^4(\tilde{L}) := I_b^1(\tilde{L}) \cap (L_4, \infty)$.

Remark 5 *Theorem 2 shows that for* $\mathfrak{v} = 0$ *the ambiguity in defining the semi travel time of homoclinic solutions from a neighborhood of* $q_1(0)$ *to* $q_2(0)$ *is exponentially small in* L

The fact that all 1–homoclinic orbits near the heteroclinic cycle consisting of q_1 *and* q_2 *are contained in the family* $h_{L,0}$ *is a local uniqueness of these 1–homoclinic orbits up to time shifts. In remark 4 this is expressed more intuitively using sections* Σ_1*,* Σ_2*.*

By definition of I_b^4*, there is* $j_1 \geq j_0$ *such that the countable family of curves of 1–homoclinic orbits can be parametrized as* $h_{j,r} := h_{T_j + r, 0}$ *where* $j \in \mathbb{N}$*,* $j \geq j_1$ *and* $|r| < \mathrm{e}^{-\kappa \sup I_j}$*. With this parametrization,* j *counts the number of times the 1–homoclinic winds around* γ*, e.g.* $h_{j+1,0}$ *has one more 'hump' than* $h_{j,0}$*.*

The fact that I_b^4 *is a disconnected set implies that for* $\mathfrak{v} = 0$ *the set of parameter values* $\{\mu(L, \mathfrak{v}) | L \in I_b^1 \cap (L_4, \infty)\}$ *consists of a union of disjoint curve segments near* $\mu = 0$*, i.e. if* $\mu(L, 0) = \mu(L', 0)$ *for* $L \leq L' \in I_b^4$*, then* $L = L'$*. This follows locally from the implicit function theorem and hypothesis 4, and globally from* $B_{\epsilon_4}(0; L') \subset B_{\epsilon_4}(0; L)$ *for* $L \leq L'$ *and the uniqueness up to time shifts of 1–homoclinic orbits in these balls.*

The disjoint parameter curves may be connected to form a smooth curve with parameter values for homoclinic orbits beyond the reach of theorem 2. In section 4 we consider this for the codimension–2 case analytically by essentially pathfollowing solutions in $\mathfrak{v}$. Numerically this has been found even for the codimension–1 case in [37].

2a) Matching near the transverse heteroclinic

As a first step, we match $q_1(0)+w_1^1(W_0,\mu,L)(0)$ with the unstable manifold of p_0 using the exponential dichotomies of the variation about $q_1(\xi)$ for $\xi \le 0$, see lemma 1. The unstable manifold of p_0 near $q_1(0)$ is a graph over $q_1(0)+E^{\mathrm{u}}_{-1}(0)$, and there exists $\epsilon_{\mathrm{u}} > 0$ and a C^k-function $m_1(\cdot;\mu) : E^{\mathrm{u}}_{-1}(0) \to E^{\mathrm{s}}_{-1}(0)$ such that for $|v| < \epsilon_{\mathrm{u}}$, and $\mu \in \Lambda$ the point $q_1(0)+v+m_1(v;\mu) \in \mathcal{W}^{\mathrm{u}}(p_0;\mu)$, cf. e.g. [7]. We possibly have to shrink Λ for this, keeping an open set. Furthermore, $m_1(0;0) = 0$, $\partial_v m_1(0;0) = 0$ and for a suitable constant K_6 we have

$$|m_1(v;\mu)| \le K_6(|v|^2 + |\mu|). \tag{3.29}$$

Therefore, matching a glued solution $W^1 = (w_1^1, w_2^1)$ from theorem 1 with the unstable manifold near the heteroclinic q_1 means to solve the equation

$$w_1^1(W_0,\mu,L)(0) = v + m_1(v;\mu). \tag{3.30}$$

This will be achieved in terms of $\tilde{w}_1$ and v, leaving $\tilde{w}_2$, μ, and L as parameters to match near the codimension–1 or –2 heteroclinic q_2.

Lemma 6 *Assume hypotheses 1, and 2 or 3. There exist positive constants C, L_2, ϵ_2, δ_2 such that for all $\tilde{L} \in I_b^1 \cap (L_2,\infty)$, there exist C^k functions $\tilde{w}_1(\tilde{w}_2,\mu,L)$ and $\tilde{v}(\tilde{w}_2,\mu,L,\mathfrak{v})$, for $|\tilde{w}_2|+\mathrm{e}^{\eta\tilde{L}}|\mu|+|\mathfrak{v}| < \delta_2$, $L \in I_b^1(\tilde{L})\cap(L_2,\infty)$, which solve (3.30) with $W_0 = (\tilde{w}_1(\tilde{w}_2,\mu,L,\mathfrak{v}),\tilde{w}_2)$ and $v = \tilde{v} + \mathfrak{v}$. These are the unique solutions to (3.30) with $|v| + |\tilde{w}_1| < \epsilon_2$ for any $\tilde{w}_2,\mu,L,\mathfrak{v}$ as above.*

Let u be a solution to (2.1) with variations $W(\sigma,L) \in B^0_{\epsilon_2}(0;L)$ which solves (3.30) for $|w^{\mathrm{u}}_{-2}(0;\sigma_{\ell_L},\ell_L)| + |\tilde{Q}w^{\mathrm{u}}_{-1}(0;\sigma_{\ell_L})| + |\mu| < \delta_2$. Then $W_0(\sigma_{\ell_L},\ell_L)$ satisfies

$$|w_1(0;\sigma_{\ell_L})| \le C\left(|w^{\mathrm{u}}_{-2}(0;\sigma_{\ell_L},\ell_L)| + |\mu| + \mathrm{e}^{2(\eta-\kappa)L} + |\tilde{Q}w^{\mathrm{u}}_{-1}(0;\sigma_{\ell_L})|\right).$$

Proof. Theorem 1 applies and yields glued solutions $W^1(W_0,\mu,L)$ for all $|W_0| < \delta$, $|\mu| < \delta\mathrm{e}^{-\eta L}$, $L \in I_b^1$ which satisfy

$$W^1(W_0,\mu,L)(0) = \mathcal{A}(L)(0)W_0 + \mathcal{N}(W^1(W_0,\mu,L);\mu,L)(0) + \mathcal{B}(L)(0)$$

and the first component w_1^1 as defined in (3.16, 3.18, 3.19) at $\xi = 0$ is

$$\begin{aligned}
\mathcal{A}_1(L)(0)W_0 &= \tilde{w}_1 + \Phi^{\mathrm{u}}_{+1}(0,L)P^{\mathrm{u}}_L\mathfrak{c}_1(W_0,L)\\
\mathcal{N}_1(W^1(W_0,\mu,L);\mu,L)(0) &= \textstyle\int_L^0 \Phi^{\mathrm{cu}}_{+1}(0,\zeta)g_1(w_1^1(W_0,\mu,L)(\zeta),\zeta;\mu)\mathrm{d}\zeta+\\
&\quad +\Phi^{\mathrm{u}}_{+1}(0,L)P^{\mathrm{u}}_L\mathfrak{c}_2(W^1(W_0,\mu,L);L,\mu)\\
\mathcal{B}_1(L)(0) &= \Phi^{\mathrm{u}}_{+1}(0,L)P^{\mathrm{u}}_L b_L.
\end{aligned}$$

With the above notation, the matching equation (3.30) can be viewed as a nonlinear perturbation of $v - \tilde{w}_1 = e$, $e \in \mathbb{R}^n$, $v \in E^{\mathrm{u}}_{-1}(0)$, $\tilde{w} \in E^{\mathrm{s}}_{+1}(0)$. This linear equation is generally solvable by both hypothesis 2 or 3, and the solution is unique up to a component in $E^{\mathrm{u}}_{-1}(0) \cap E^{\mathrm{s}}_{+1}(0)$. We project by $\mathrm{Id} - \tilde{Q}$ to a complement $\tilde{E}^{\mathrm{u}}_{-1}$ of $E^{\mathrm{u}}_{-1}(0) \cap E^{\mathrm{s}}_{+1}(0)$ in $E^{\mathrm{u}}_{-1}(0)$, so that $\mathbb{R}^n = \tilde{E}^{\mathrm{u}}_{-1} \oplus E^{\mathrm{s}}_{+1}(0)$ and view $\mathfrak{v} := \tilde{Q}v$ as a parameter. For any $\mathfrak{v} \in E_1$ the map $\tilde{E}^{\mathrm{u}}_{-1} \times E^{\mathrm{s}}_{+1}(0) \to \mathbb{R}^n; (\tilde{v},\tilde{w}) \mapsto \tilde{v} - \tilde{w}_1 + \mathfrak{v}$ is invertible. Since $\mathfrak{c}_1(W_0,L)$ is linear in W_0 equation (3.30) contains a perturbed linear map $D^1_L(\tilde{v},\tilde{w}_1) := \tilde{v} - \tilde{w}_1 - \Phi^{\mathrm{u}}_{+1}(0,L)P^{\mathrm{u}}_L\Phi^{\mathrm{s}}_{+1}(L,0)\tilde{w}_1$. If $L \in I^1_b \cap (\tilde{L}_2,\infty)$, for sufficiently large $\tilde{L}_2$, then D^1_L is invertible with uniformly bounded norm $C_1 > 0$ of the inverse, because by (2.7)

$$|\Phi^{\mathrm{u}}_{+1}(0,L)P^{\mathrm{u}}_L\Phi^{\mathrm{s}}_{+1}(L,0)\tilde{w}_1| \le C\mathrm{e}^{-2\kappa L}|\tilde{w}_1|.$$

(By remark 2 and the definition of I^1_b the rate is $-3\kappa L$.) We write (3.30) as

$$\begin{aligned}
D^1_L(\tilde{v},\tilde{w}_1) = &\mathcal{N}_1(W^1((\tilde{w}_1,\tilde{w}_2),\mu,L);\mu,L)(0) + \mathcal{B}_1(L)(0) - m_1(\tilde{v}+\mathfrak{v};\mu)\\
&-\Phi^{\mathrm{u}}_{+1}(0,L)P^{\mathrm{u}}_L\Phi^{\mathrm{u}}_{-2}(-L,0)\tilde{w}_2 - \mathfrak{v}.
\end{aligned}$$

Using lemma 5 and (3.13) there is a constant $C_2 > 0$ such that for $\tilde{L} \in I^1_b \cap (\tilde{L}_2,\infty)$ and $L \in I^1_b(\tilde{L})$ we have

$$\begin{aligned}
|\mathcal{N}_1(W^1((\tilde{w}_1,\tilde{w}_2),\mu,L);\mu,L)(0)| &\le C_2(||W^1||^2_{\tilde{L}} + |\mu|\,||W^1||_{\tilde{L}} + |\mu|)\\
&\le C_2\left(|(\tilde{w}_1,\tilde{w}_2)|(|(\tilde{w}_1,\tilde{w}_2)| + |\mu| + \mathrm{e}^{(\eta-\kappa)\tilde{L}}) + |\mu| + \mathrm{e}^{2(\eta-\kappa)\tilde{L}}\right),
\end{aligned}$$

and analogously for the derivative of $\mathcal{N}_1(W^1,\mu,L)$ with respect to $\tilde{w}_1$ at $\xi = 0$. Including the linear term from $\mathcal{N}_1(W^1)$, the linear part of (3.30) is of the form $D^1_L + O(\mathrm{e}^{(\eta-\kappa)L} + |\mu|)$. There exist constants $\bar{L}_2 \ge \tilde{L}_2$ and $0 < \bar{\delta} \le \min\{\epsilon_{\mathrm{u}},\delta\}$, where $\delta > 0$ is from theorem 1, such that for all $L \in I^1_b \cap (\bar{L}_2,\infty)$ and $|\mu| < \bar{\delta}$ this linear map is invertible with uniform bounded norm $C_1 > 0$ of the inverse. By virtue of (2.7) and lemma 3, we can estimate

$$|\mathcal{B}_1(L)(0)| \le C\mathrm{e}^{-2\kappa L},$$

and therefore choose $L_2 \ge \bar{L}_2$ so that $|\mathcal{B}_1(L)(0)| + C_2\mathrm{e}^{2(\eta-\kappa)L} \le 1/(8C_1^2C_2)$. The implicit function theorem proposition 1 applies with the following choices for any $\tilde{L} \in I_b^1 \cap (\bar{L}_2, \infty)$: $X = \mathbb{R}^n$, $Y = B_{\delta_*}(0) \subset \tilde{E}^{\mathrm{u}}_{-1} \times E^{\mathrm{s}}_{+1}(0)$, $y = (\tilde{w}_1, \tilde{v})$. For $B_{\delta_*}(0) \subset E^{\mathrm{u}}_{-2}(0) \times E_1$ define $Z = B_{\delta_*}(0) \times \Lambda^{\delta_*}_{\tilde{L}}$, $z = (\tilde{w}_2, \mathfrak{v}, \mu)$, and $I := I_b^1(\tilde{L}) \cap (L_2, \infty)$. Using the terms contained in $\mathcal{N}_1$ we set $Q(z, L) := D_L^1 + O(\mathrm{e}^{(\eta-\kappa)L})$, $R(y, z, L) := m_1(\tilde{v} + \mathfrak{v}; \mu) + O(|z| + |y|(|y| + |z|))$, and $S(L) := \mathcal{B}_1(L)(0) + O(\mathrm{e}^{2(\eta-\kappa)L})$. As in the proof of theorem 1, all constants are independent of $\tilde{L}$, hence the estimates and constants from proposition 1 are.

Hence, there are constants ϵ', δ_2 and unique solutions $(\tilde{v}, \tilde{w}_1)(\tilde{w}_2, \mu, L, \mathfrak{v}) \in B_{\epsilon'}(0) \subset Y$, where $\tilde{v}$, $\tilde{w}_1$ are C^k in L, μ, $\mathfrak{v}$ and $\tilde{w}_2$ for all $\tilde{L} \in I_b^1 \cap (L_2, \infty)$, $L \in I_b^1(\tilde{L}) \cap (L_2, \infty)$ and $\mathrm{e}^{\eta\tilde{L}}|\mu| + |\tilde{w}_2| + |\mathfrak{v}| < \delta_2$. In addition, proposition 1 and theorem 1 imply that the following estimate also holds for $|\mu| + |\tilde{w}_2| + |\mathfrak{v}| < \delta_2$:

$$
\begin{aligned}
|\tilde{w}_1(\tilde{w}_2, \mu, L, \mathfrak{v})| + |\tilde{v}(\tilde{w}_2, \mu, L, \mathfrak{v})| &\leq C\left(|\tilde{w}_2| + |\mu| + \mathrm{e}^{2(\eta-\kappa)L} + \mathrm{e}^{-2\kappa L} + |\mathfrak{v}|\right) \\
&\leq C\left(|\tilde{w}_2| + |\mu| + \mathrm{e}^{2(\eta-\kappa)L} + |\mathfrak{v}|\right).
\end{aligned}
$$

Let $W(\sigma, L)$ be the vector of variation for a given solution and let ϵ be from theorem 1. For any $\epsilon_2 \leq \min\{\epsilon', \epsilon\}$, the unique solution obtained from $W(\sigma, L) \in B^0_{\epsilon_2}(0; L)$ by theorem 1 is $W^1(W_0(\sigma_{\ell_L}, \ell_L), \mu, \ell_L)$. For ϵ_2 sufficiently small the smoothness then implies $|w^{\mathrm{s}}_{+1}(0; \sigma_{\ell_L})| + |w^{\mathrm{u}}_{-1}(0; \sigma_{\ell_L})| < \epsilon'$. Hence, these coincide with the above solutions for $\mathfrak{v} = \tilde{Q} w^{\mathrm{u}}_{-1}(0; \sigma_{\ell_L})$, if $\mu \in \Lambda^{\delta_2}_{\tilde{L}}$. $\square$

We now have obtained variational solutions $w_1^1((\tilde{w}_1(\tilde{w}_2, \mu, L, \mathfrak{v}), \tilde{w}_2), \mu, L) \in C_{\eta,\tilde{L}}$ which make solutions that lie in the unstable manifold of the equilibrium p_0 and pass close to the periodic orbit γ. Note that existence and local uniqueness only hold for $|\mu| < \delta_2 \mathrm{e}^{-\eta L}$, but the estimate holds already for such solutions if $|\mu| < \delta_2$. We denote $I_b^2 := I_b^1 \cap (L_2, \infty)$, $I_b^2(\tilde{L}) := I_b^1(\tilde{L}) \cap (L_2, \infty)$,

$$
\begin{aligned}
W_0^2(\tilde{w}_2, \mu, L, \mathfrak{v}) &:= (\tilde{w}_1(\tilde{w}_2, \mu, L, \mathfrak{v}), \tilde{w}_2) \\
W^2(\tilde{w}_2, \mu, L, \mathfrak{v}) &:= W^1(W_0^2(\tilde{w}_2, \mu, L, \mathfrak{v}), \mu, L).
\end{aligned}
$$

From (3.13) and lemma 6 we conclude the estimates

$$
\begin{aligned}
|W_0^2(\tilde{w}_2, \mu, L, \mathfrak{v})| &\leq C\left(|\tilde{w}_2| + |\mu| + \mathrm{e}^{2(\eta-\kappa)L} + |\mathfrak{v}|\right) \\
||W^2(\tilde{w}_2, \mu, L, \mathfrak{v})||_{\tilde{L}} &\leq C\left(|\tilde{w}_2| + |\mu| + \mathrm{e}^{(\eta-\kappa)\tilde{L}} + |\mathfrak{v}|\right).
\end{aligned}
\tag{3.31}
$$

2b) Match near the codimension–d heteroclinic, Ljapunov-Schmidt reduction

To complete the homoclinic to p_0, we want to find $\tilde{w}_2$, μ and L such that $w_1^2(\tilde{w}_2, \mu, L, \mathfrak{v}) \in \mathcal{W}^{\mathrm{s}}(p_0; \mu)$. Analogous to the previous step, we use the dichotomy $P^{\mathrm{s}}_{+2}(\xi)$ and $P^{\mathrm{u}}_{+2}(\xi)$ from lemma 1 for the variation about q_2 towards p_0. The stable manifold near $q_2(0)$ is a graph over $q_2(0)+E^{\mathrm{s}}_{+2}(0)$ given by a C^k function $m_2(\cdot;\mu) : B_{\epsilon_{\mathrm{s}}}(0) \subset E^{\mathrm{s}}_{+2}(0) \to E^{\mathrm{u}}_{+2}(0)$ with suitable ϵ_{s}, and for $\mu \in \Lambda$, possibly shrunken, and it satisfies, cf. [7], the estimate

$$|m_2(v;\mu)| \le C(|v|^2 + |\mu|). \tag{3.32}$$

As before, this allows to formulate matching through the equation

$$w^2_2(\tilde{w}_2,\mu,L,\mathfrak{v})(0) = v + m_2(v;\mu) \tag{3.33}$$

where the left hand side satisfies

$$\begin{aligned} w^2_2(\tilde{w}_2,\mu,L,\mathfrak{v})(0) = {} & \mathcal{A}_2(L)(0)W^2_0(\tilde{w}_2,\mu,L,\mathfrak{v}) \\ & +\mathcal{N}_2(W^2(\tilde{w}_2,\mu,L,\mathfrak{v});\mu,L)(0) + \mathcal{B}_2(L)(0) \end{aligned}$$

and from (3.16, 3.18, 3.19) at $\xi = 0$ the details are

$$\begin{aligned} \mathcal{A}_2(L)(0)W^2_0(\tilde{w}_2,\mu,L,\mathfrak{v}) &= \tilde{w}_2 - \Phi^{\mathrm{sc}}_{-2}(0,-L)P^{\mathrm{sc}}_L\mathfrak{c}_1(W^2_0(\tilde{w}_2,\mu,L,\mathfrak{v}),L) \\ \mathcal{N}_2(W^2(\tilde{w}_2,\mu,L,\mathfrak{v});\mu,L)(0) &= \int_{-L}^{0} \Phi^{\mathrm{sc}}_{-2}(0,\zeta)g_2(w^2_2(\tilde{w}_2,\mu,L,\mathfrak{v}),\zeta;\mu)\mathrm{d}\zeta - \\ &\quad - \Phi^{\mathrm{sc}}_{-2}(0,-L)P^{\mathrm{sc}}_L\mathfrak{c}_2(W^2(\tilde{w}_2,\mu,L,\mathfrak{v}),L) \quad (3.34) \\ \mathcal{B}_2(L)(0) &= -\Phi^{\mathrm{sc}}_{-2}(0,-L)P^{\mathrm{sc}}_L b_L \end{aligned}$$

We will see below, that the linear part of (3.33) is a small perturbation of $\tilde{w}_2 - v$. By hypothesis 2 or 3, $\dim(E^{\mathrm{s}}_{+2}(0)+E^{\mathrm{u}}_{-2}(0)) = n-d$, whence we use Ljapunov-Schmidt reduction to solve (3.33) first for $(v,\tilde{w}_2)$ in $E_2 := E^{\mathrm{s}}_{+2}(0)+E^{\mathrm{u}}_{-2}(0) \sim \mathbb{R}^{n-d}$ and then in the complement $(E_2)^\perp \sim \mathbb{R}^d$. Let P_2 denote the projection $P^{\mathrm{u}}_{-2}(0)+P^{\mathrm{s}}_{+2}(0)$ onto E_2 with kernel $(E_2)^\perp$ in $\mathbb{R}^n$ and $\kappa' := \min\{2(\kappa-\eta),\kappa\}$.

Lemma 7 *Assume hypothesis 1, and 2 or 3. There exist strictly positive constants ϵ_3, L_3 and δ_3, such that for all $\tilde{L} \in I^2_b \cap (L_3,\infty)$, there exist smooth functions $\tilde{w}_2(\mu,L,\mathfrak{v})$ and $v(\mu,L,\mathfrak{v})$ for $|\mu| < \mathrm{e}^{-\eta\tilde{L}}\delta_3$, $L \in I^2_b(\tilde{L}) \cap (L_3,\infty)$ and $|\mathfrak{v}| \le \delta_3$, which solve (3.33) in E_2, i.e.*

$$P_2 w^2_2(\tilde{w}_2(\mu,L,\mathfrak{v}),\mu,L)(0) = v(\mu,L,\mathfrak{v}) + P_2 m_2(v(\mu,L,\mathfrak{v});\mu)).$$

These are the unique solutions in $B_{\epsilon_3}(0) \subset E_2$. Let u be a solution to (2.1) with variations $W(\sigma,L) \in B^0_{\epsilon_3}(0;L)$ which solve (3.30, 3.33) for some $|\mu| + |\tilde{Q}w^{\mathrm{u}}_{-1}(0;\sigma_{\ell_L})| < \delta_3$. Then

$$|w^{\mathrm{u}}_{-2}(0;\sigma_{\ell_L})| + |w^{\mathrm{s}}_{+2}(0;\sigma_{\ell_L},\ell_L)| \le C(|\mu| + \mathrm{e}^{-\kappa'\tilde{L}} + |\tilde{Q}w^{\mathrm{u}}_{-1}(0;\sigma_{\ell_L})|).$$

Proof. By the assumptions theorem 1 and lemma 6 apply. In E_2 the linear map $D_L^2 : E_{+2}^{\mathrm{s}}(0) \times E_{-2}^{\mathrm{u}}(0) \to E_2 \sim \mathbb{R}^{n-d}$, $(v, \tilde{w}_2) \mapsto v - \tilde{w}_2$ is invertible by both hypothesis 2 or 3. In E_2, (3.33) is of the form

$$\begin{aligned} D_L^2(v, \tilde{w}_2) = P_2 \Big(& -\Phi_{-2}^{\mathrm{sc}}(0, -L) P_L^{\mathrm{sc}} \mathfrak{c}_1(W_0^2(\tilde{w}_2, \mu, L, \mathfrak{v}), L) + \\ & + \mathcal{N}_2(W^2(\tilde{w}_2, \mu, L, \mathfrak{v}); \mu, L)(0) + \mathcal{B}_2(L)(0) - m_2(v; \mu) \Big) . \end{aligned}$$

Using the definition of I_b, remark 2, (2.7) and (3.31) we can estimate

$$|\Phi_{-2}^{\mathrm{sc}}(0, -L) P_L^{\mathrm{sc}} \mathfrak{c}_1(W_0^2(\tilde{w}_2, \mu, L, \mathfrak{v}), L)| \le C \mathrm{e}^{-2\kappa L}(|\tilde{w}_2| + |\mu| + \mathrm{e}^{2(\eta-\kappa)L} + |\mathfrak{v}|)$$

$$\begin{aligned} |\mathcal{N}_2(W^2(\tilde{w}_2, \mu, L, \mathfrak{v}); \mu, L)(0)| &\le C(||W^2||_{\tilde{L}}^2 + |\mu| \, ||W^2||_{\tilde{L}} + |\mu|) \\ &\le C \Big(|\tilde{w}_2|(|\tilde{w}_2| + |\mu| + \mathrm{e}^{(\eta-\kappa)\tilde{L}} + |\mathfrak{v}|) + |\mathfrak{v}|(|\mathfrak{v}| + |\mu| + \mathrm{e}^{(\eta-\kappa)\tilde{L}}) \\ &\qquad + |\mu| + \mathrm{e}^{2(\eta-\kappa)\tilde{L}} \Big) \end{aligned}$$

$$\text{and} \qquad |\mathcal{B}_2(L)(0)| \le C \mathrm{e}^{-\kappa \tilde{L}}.$$

To apply proposition 1 we proceed as in the proof of lemma 6 and omit some details.

The parts of these estimates linear in $|\tilde{w}_2|$ are $O(\mathrm{e}^{(\eta-\kappa)\tilde{L}} + |\mathfrak{v}| + |\mu|)$, and we find $L_3 \ge L_2$, $\bar{\delta}_3 \le \delta_2$ such that for $L \ge L_3$ and $|\mathfrak{v}| + |\mu| < \bar{\delta}_3$ the perturbed linear map $D_L^2 + O(\mathrm{e}^{(\eta-\kappa)L} + |\mathfrak{v}| + |\mu|)$ is invertible with uniformly bounded norm of the inverse.

Set $\epsilon' := \min\{\epsilon_2, \bar{\delta}_3\}$ and $\delta' := \min\{\epsilon_{\mathrm{s}}, \bar{\delta}_3\}$. Together with (3.32), the above estimates allow to apply proposition 1 for any $\tilde{L} \in I_b^2 \cap (L_3, \infty)$ with $I := I_b^2(\tilde{L})$, $X = E_2$, $Y = B_{\epsilon'}(0) \subset E_2$ and $Z = \Lambda_{\tilde{L}}^{\delta'} \times B_{\delta'}(0) \subset \Lambda \times E_1$. So $y = v$, $z = (\mu, \mathfrak{v})$ and we take $Q(z, L) := D_L^2 + O(\mathrm{e}^{(\eta-\kappa)L} + |\mathfrak{v}| + |\mu|)$, $R(y, z, L) := m_2(v; \mu) + O(|z| + |y|(|z| + |y|))$, and $S(L) := \mathcal{B}_2(L)(0) + O(\mathrm{e}^{2(\eta-\kappa)\tilde{L}})$. Again all constants are uniform in $\tilde{L}$, so are the resulting constants and estimates.

We obtain constants $\epsilon_3 > 0$, $\delta_3 > 0$, C and unique solutions $(v, \tilde{w}_2) = (v, \tilde{w}_2)(\mu, L, \mathfrak{v}) \in B_{\epsilon_3}(0) \subset E_2$ which are C^k functions for $\mathrm{e}^{\eta \tilde{L}}|\mu| + |\mathfrak{v}| < \delta_3$, and $\tilde{L} \in I_b^2 \cap (L_3, \infty)$, $L \in I_b^2(\tilde{L}) \cap (L_3, \infty)$ and for $|\mu| < \delta_3$ these satisfy

$$|\tilde{w}_2| + |v| \le C \left(|\mu| + \mathrm{e}^{2(\eta-\kappa)\tilde{L}} + \mathrm{e}^{-\kappa \tilde{L}} + |\mathfrak{v}| \right) \le C \left(|\mu| + \mathrm{e}^{-\kappa' L} + |\mathfrak{v}| \right).$$

The local uniqueness statement follows analogous to the one in lemma 6 by possibly decreasing $\epsilon_3 > 0$. □

Thus, we obtained glued solution, matched everywhere except in a complement of E_2 and denote $I_b^3 := I_b^2 \cap (L_3, \infty)$, $I_b^3(\tilde{L}) := I_b^2(\tilde{L}) \cap (L_3, \infty)$ and

$$
\begin{aligned}
W_0^3(\mu,L,\mathfrak{v}) &:= W_0^2(\tilde{w}_2(\mu,L,\mathfrak{v}),\mu,L)\\
W^3(\mu,L,\mathfrak{v}) &:= W^2(\tilde{w}_2(\mu,L,\mathfrak{v}),\mu,L).
\end{aligned}
$$

From (3.13) and lemma 7 we conclude the estimates

$$
\begin{aligned}
|W_0^3(\mu,L,\mathfrak{v})| &\leq C\left(|\mu| + \mathrm{e}^{-\kappa' L} + |\mathfrak{v}|\right)\\
||W^3(\mu,L,\mathfrak{v})||_{\tilde{L}} &\leq C\left(|\mu| + \mathrm{e}^{(\eta-\kappa)\tilde{L}} + |\mathfrak{v}|\right).
\end{aligned} \tag{3.35}
$$

2c) Match in complement, Melnikov's method, proof of theorem 2

To solve (3.33) in the d-dimensional complement $(E_2)^\perp$, we use the basis $\{a_0^1, a_0^2\} \subset \mathbb{R}^n$ of $(E_2)^\perp$ introduced at the beginning of this section. Throughout this subsection j takes on both values $j=1$ and $j=2$. The matching is complete, if

$$
\langle w_2^3(\mu,L,\mathfrak{v})(0), a_0^j\rangle = \langle m_2(v(\mu,L,\mathfrak{v});\mu), a_0^j\rangle. \tag{3.36}
$$

Let $\tilde{L} \in I_b^3$, $L \in I_b^3(\tilde{L})$. Again, we exploit the fixed point equation

$$
w_2^3(\mu,L,\mathfrak{v})(0) = \mathcal{A}_2(L)(0)W_0^3(\mu,L,\mathfrak{v}) + \mathcal{N}_2(w_2^3(\mu,L,\mathfrak{v}),\mu)(0) + \mathcal{B}_2(L)(0)
$$

with details as in (3.34). Since $\tilde{w}_2(\mu,L,\mathfrak{v}) \in E_2$, it follows $\langle \tilde{w}_2(\mu,L,\mathfrak{v}), a_0^j\rangle = 0$ and $\dot{q}_2(0) \in E^{\mathrm{s}}_{+2}(0) \subset E_2$, so $\langle P^{\mathrm{c}}_{-2}(0)\cdot, a_0^j\rangle \equiv 0$. Hence, the center direction is not visible in this matching and (3.36) is in fact equivalent to

$$
\begin{aligned}
&\langle \int_{-L}^0 \Phi^{\mathrm{s}}_{-2}(0,\zeta) g_2(w_2^3(\mu,L,\mathfrak{v})(\zeta),\zeta;\mu)\mathrm{d}\zeta - m_2(v(\mu,L,\mathfrak{v});\mu), a_0^j\rangle \\
&= -\langle \Phi^{\mathrm{s}}_{-2}(0,-L)P^{\mathrm{s}}_L\left(\mathfrak{c}_1(W_0^3(\mu,L,\mathfrak{v}),L) + \mathfrak{c}_2(W^3(\mu,L,\mathfrak{v}),L) + b_L\right), a_0^j\rangle.
\end{aligned} \tag{3.37}
$$

For later reference, we single out the expected leading order term in L on the right hand side. From the estimates (3.21), (3.25), (3.26) and (3.35), we obtain

$$
\begin{aligned}
&|\langle \Phi^{\mathrm{s}}_{-2}(0,-L)P^{\mathrm{s}}_L\left(\mathfrak{c}_1(W_0^3(\mu,L,\mathfrak{v}),L) + \mathfrak{c}_2(W^3(\mu,L,\mathfrak{v}),L)\right), a_0^j\rangle|\\
&\qquad\leq \mathrm{e}^{-\kappa L}\Big(\mathrm{e}^{-\kappa L}(|\mu| + \mathrm{e}^{-\kappa' L} + |\mathfrak{v}|)\\
&\qquad\quad + \mathrm{e}^{-\eta L}(\mathrm{e}^{2(\eta-\kappa)L} + |\mathfrak{v}|(|\mathfrak{v}| + |\mu| + \mathrm{e}^{(\eta-\kappa)L}) + |\mu|)\Big)\\
&\qquad\leq \mathrm{e}^{(\eta-3\kappa)L} + \mathrm{e}^{-\kappa L}|\mathfrak{v}| + \mathrm{e}^{-(\kappa+\eta)L}\left(|\mathfrak{v}|(|\mathfrak{v}| + |\mu|) + |\mu|\right).
\end{aligned} \tag{3.38}
$$

The full right hand side of (3.37) thus can be written as

$$
-\langle \Phi^{\mathrm{s}}_{-2}(0,-L)P^{\mathrm{s}}_L b_L, a_0^j\rangle + O\left(\mathrm{e}^{(\eta-3\kappa)L} + \mathrm{e}^{-\kappa L}|\mathfrak{v}| + \mathrm{e}^{-(\kappa+\eta)L}|\mu|\right) \tag{3.39}
$$

Since $0 < \eta < \kappa$ it holds that $\eta - 3\kappa < -2\kappa$ and so the leading order term in L is expected to be $\langle \Phi^{\mathrm{s}}_{-2}(0,-L)P^{\mathrm{s}}_L b_L, a_0^j\rangle$.

For the left hand side of (3.37), we set up Melnikov-type integrals. Since the image of the trichotomy projection $P^{\mathrm{s}}_{-2}(0)$ is arbitrary, as long as trivially intersecting $\mathrm{Rg}(P^{\mathrm{u}}_{-2}(0))$, cf. [8], we may assume for the adjoint projection $(E_2)^\perp \subset \mathrm{Rg}((P^{\mathrm{s}}_{-2}(0))^*)$. Hence, we have $a_0^j = (P^{\mathrm{s}}_{-2}(0))^* a_0^j$ and

$$a^j(\zeta) = (\Phi^{\mathrm{s}}_{-2}(0,\zeta))^* a_0^j = (\Phi_2(0,\zeta))^*(P^{\mathrm{s}}_{-2}(0))^* a_0^j = (\Phi_2(0,\zeta))^* a_0^j.$$

Inspecting the linearization of the left hand side of (3.37) with respect to μ at $\mu = 0$, recall the definition of g_2 in (2.5), we obtain on the one hand for $j = 1, 2$ that

$$\langle \int_{-L}^0 \Phi^{\mathrm{s}}_{-2}(0,\zeta) f_\mu(q_2(\zeta) + w_2^3(0, L, \mathfrak{v})(\zeta); 0)\mu \mathrm{d}\zeta, a_0^j\rangle = \\ \int_{-L}^0 \langle f_\mu(q_2(\zeta) + w_2^3(0, L, \mathfrak{v})(\zeta); 0)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta.$$

From (3.35), $||w_2^3(0, L, \mathfrak{v})||_{-\eta} \le C(\mathrm{e}^{(\eta-\kappa)\tilde{L}} + |\mathfrak{v}|)$ and, since the trichotomy estimates hold for the adjoint equation, it follows $|a^j(\zeta)| \le C\mathrm{e}^{\kappa\zeta}|a_0^j|$ for $\zeta \le 0$. Hence, for $\tilde{L} \in I_b^3$, $L \in I_b^3(\tilde{L})$ we can approximate by the w_2^3 independent integral

$$\begin{aligned}
&|\int_{-L}^0 \langle \Big(f_\mu(q_2(\zeta) + w_2^3(0, L)(\zeta); 0) - f_\mu(q_2(\zeta); 0)\Big)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta| \\
&\quad \le C \int_{-L}^0 \sup_{\substack{s\in[0,1]\\ \zeta\in[-L,0]}} |f_\mu'(q_2(\zeta) + s w_2^3(0, L, \mathfrak{v})(\zeta); 0)|\, |w_2^3(0, L, \mathfrak{v})(\zeta)|\, |\mu|\, |a^j(\zeta)| \mathrm{d}\zeta \\
&\quad \le C|\mu| \int_{-L}^0 \mathrm{e}^{\eta\zeta} ||w_2^3(0, L, \mathfrak{v})||_{-\eta,\tilde{L}} \mathrm{e}^{-\kappa\zeta}|a_0^j| \mathrm{d}\zeta \le C|\mu|\, ||w_2^3(0, L, \mathfrak{v})||_{-\eta,\tilde{L}} \\
&\quad \le C(\mathrm{e}^{(\eta-\kappa)L} + |\mathfrak{v}|)|\mu|.
\end{aligned}$$

The difference of the w_2^3 independent term to the infinite integral satisfies for $j = 1, 2$

$$\begin{aligned}
|\int_{-\infty}^{-L} \langle f_\mu(q_2(\zeta); 0)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta| &\le \int_{-\infty}^{-L} |f_\mu(q_2(\zeta); 0)\mu||a^j(\zeta)| \mathrm{d}\zeta \\
&\le C|\mu| \int_{-\infty}^{-L} |a^j(\zeta)| \mathrm{d}\zeta \le C|\mu| \mathrm{e}^{-\kappa L}
\end{aligned}$$

On the other hand, the graph m_2 satisfies, cf. e.g. [21],

$$\left(\left.\frac{\mathrm{d}}{\mathrm{d}\mu}\right|_{\mu=0} m_2(v(\mu, L); \mu) \right) \mu = -\int_0^\infty P^{\mathrm{u}}_{+2}(0)\Phi_2(0,\zeta) f_\mu(q_2(\xi); 0)\mu \mathrm{d}\zeta.$$

As above for the trichotomy near γ, we may adjust $\mathrm{Rg}(P^{\mathrm{u}}_{+2}(0))$ so that we have $(P^{\mathrm{u}}_{+2}(0))^* a_0^j = a_0^j$ and therefore

$$\int_{-\infty}^{\infty} \langle f_\mu(q_2(\zeta);0)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta = \int_{-L}^{0} \langle f_\mu(q_2(\zeta)+w_2^3(0,L,\mathfrak{v})(\zeta);0)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta$$
$$+\int_0^\infty \langle f_\mu(q_2(\xi);0)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta + O\left((\mathrm{e}^{(\eta-\kappa)L}+|\mathfrak{v}|)|\mu|\right).$$

All in all (3.37) is of the form (again $\mathfrak{v}$ in a rough estimate)

$$\begin{aligned}\int_{-\infty}^{\infty} \langle f_\mu(q_2(\zeta);0)\mu, a^j(\zeta)\rangle \mathrm{d}\zeta &= -\langle P_L^{\mathrm{s}} b_L, a^j(-L)\rangle \\ &+O\left(|\mu|(\mathrm{e}^{(\eta-\kappa)L}+|\mathfrak{v}|+|\mu|)+\mathrm{e}^{(\eta-3\kappa)L}+|\mathfrak{v}|\right)\end{aligned} \tag{3.40}$$

By hypothesis 4, the Melnikov-integral linear map $\mathcal{M}$ on the left hand side is invertible. The perturbed linear map $\mathcal{M}+O(\mathrm{e}^{(\eta-\kappa)L}+|\mathfrak{v}|)$ is invertible with uniformly bounded norm of the inverse if $L\in I_b^3\cap(\bar{L}_4,\infty)$, $|\mathfrak{v}|<\delta_{\mathfrak{v}}$ for sufficiently large $\bar{L}_4$ and small $\delta_{\mathfrak{v}}\le\delta_3$.

To obtain a fully matched solution pair, i.e. a 1–homoclinic orbit, we can apply proposition 1 in the same way as in the proof of lemma 6: We find a constant $L_4\ge\bar{L}_4$ and fix any $\tilde{L}\in I_b^3\cap(L_4,\infty)$. Then set $I:=I_b^3(\tilde{L})$, $X:=Y:=\Lambda_{\tilde{L}}^{\delta_3}(0)$, so $y=\mu$, $Z:=B_{\delta_{\mathfrak{v}}}(0)\subset E_1$, and $Q(L):=\mathcal{M}+O(\mathrm{e}^{(\eta-\kappa)\tilde{L}}+|\mathfrak{v}|)$, $R(y,L):=O(|\mu|(|\mu|+|\mathfrak{v}|)+|\mathfrak{v}|)$, and

$$S(L):=\sum_{j=1,2}\langle -b_L, a^j(-L)\rangle a_0^j + O(\mathrm{e}^{(\eta-3\kappa)L}). \tag{3.41}$$

Note $|S(L)|\le C\mathrm{e}^{-2\kappa L}$ by (3.39), and all constants are independent of $\tilde{L}$.

This yields positive constants ϵ_4, δ_4 and C, uniform in $\tilde{L}$, and a countably infinite family of C^k curves $\mu(L,\mathfrak{v})$ for all $\tilde{L}\in I_b^3\cap(L_4,\infty)$, $L\in I_b^3(\tilde{L})\cap(L_4,\infty)$, $|\mathfrak{v}|<\delta_4$, which provide the unique solutions in $\Lambda_{\tilde{L}}^{\epsilon_4}$. Now $W^4(L,\mathfrak{v}):=W^3(\mu(L,\mathfrak{v}),L,\mathfrak{v})$, $W_0^4(L,\mathfrak{v}):=W_0^3(\mu(L,\mathfrak{v}),L,\mathfrak{v})$ is a family of curves of 1–homoclinic solution $h_{L,\mathfrak{v}}$ by means of (3.8). These 1–homoclinic orbits and the parameter curve $\mu(L,\mathfrak{v})$ satisfy

$$\begin{aligned}|\mu(L,\mathfrak{v})| &\le C(\mathrm{e}^{-2\kappa L}+|\mathfrak{v}|)\\ |W_0^4(L,\mathfrak{v})| &\le C(\mathrm{e}^{-\kappa' L}+|\mathfrak{v}|)\\ ||W^4(L,\mathfrak{v})||_{\tilde{L}} &\le C(\mathrm{e}^{(\eta-\kappa)\tilde{L}}+|\mathfrak{v}|)\end{aligned}$$

and in particular $|h_{L,\mathfrak{v}}(0)-q_1(0)|+|h_{L,\mathfrak{v}}(2L)-q_2(0)|\le C(\mathrm{e}^{(\eta-\kappa)L}+|\mathfrak{v}|)$.

The local uniqueness in the statement of theorem 2 follows from its analogue in theorem 1, the uniqueness statements in lemmas 6, 7 and for $\mu(L,\mathfrak{v})$ above by possibly decreasing $\epsilon_4>0$. The estimate $|\mu|\le C(\mathrm{e}^{-2\kappa L}+|\mathfrak{v}|)$ follows from proposition 1 already without the existence proof, similar to the estimates in lemmas 6, 7. Hence, while the existence proof needed $|\mu| \leq \delta_4 \mathrm{e}^{-\eta L}$, the local uniqueness for 1–homoclinic orbits holds for example if $|\mathfrak{v}| \leq C\mathrm{e}^{-\kappa L}$ and $\mu \in B_{\delta_4}(0) \subset \Lambda$. Taken together we obtain the claimed uniqueness up to time shifts in $B_{\delta_4}(0) \subset \Lambda$ for $\mathfrak{v} = 0$, i.e. uniform in L. □

Recall that the set I_b^3 consists of disconnected open intervals, see remark 5, and $h_{L,\mathfrak{v}}$, $\mu(L, \mathfrak{v})$ consist of disjoint curve segments for any fixed small $\mathfrak{v}$. In section 4 the role of $\mathfrak{v}$ in connecting these pieces is investigated.

4 Extending the curves of 1–homoclinic orbits

In the codimension–2 case the 1–homoclinic orbits found in theorem 2 have the additional parameter $\mathfrak{v}$ due to the two-dimensionality of $E^{\mathrm{u}}_{-1}(0) \cap E^{\mathrm{s}}_{+1}(0)$ and $\mathcal{W}_1 = \mathcal{W}^{\mathrm{u}}(p_0) \cap \mathcal{W}^{\mathrm{cs}}(\gamma)$. This creates a one-parameter family of distinct *heteroclinic* orbits for $\mu = 0$ in $\mathcal{W}_1$, i.e. not related by time shifts. Theorem 2 may be applied to any one of these heteroclinic orbits with asymptotic phase zero, if all other hypotheses hold. In the following, we will reparametrize $h_{L,\mathfrak{v}}$ in terms of these distinct heteroclinic orbits and draw global conclusions about connected curves of 1–homoclinic orbits.

Let $q_{1,\alpha}$ be a heteroclinic orbit from p_0 to γ such that $q_{1,\alpha}(0) \in \mathcal{W}^{\mathrm{ss}}_\alpha(\gamma)$. Then $q_{1,\alpha}(\cdot - \alpha)$ has asymptotic phase zero and under the assumptions of theorem 2 we obtain a family of 1–homoclinic orbits, which we denote by $h^\alpha_{L,\mathfrak{v}}$.

Notation *For an interval J we define $\Gamma_J := \{q_{1,\alpha}(0) \in \mathcal{W}^{\mathrm{ss}}_\alpha(\gamma) \,|\, \alpha \in J\}$. For convenience, we denote $\mu(L, \alpha) := \mu(L, \alpha, 0)$ and $h_{L,\alpha} := h^\alpha_{L,0}(\cdot + \alpha)$ for $\alpha \in J$, and we also use β instead of α. (We will show that for small parameters this is a reparametrization of $h_{L,\mathfrak{v}}$ from theorem 2.)*

Let $\Phi(\xi)$ denote the flow to $\dot{u} = f(u; 0)$, cf. (2.1), and the orbit of a set $S \subset \mathbb{R}^n$ is $\mathcal{O}(S) := \{\Phi(\xi)v \,|\, v \in S, \xi \in \mathbb{R}\}$.

Lemma 8 *Assume hypothesis 1, 2 and 4. There exists a C^k curve $\Gamma \subset \mathcal{W}_1$ which contains $q_1(0)$ and is transverse to the flow and strong stable fibers of γ. Any such bounded curve Γ can be parametrized so that $\Gamma = \Gamma_J$ for a bounded nontrivial interval $J \subset \mathbb{R}$.*

Proof. By hypothesis 2 the intersection of unstable and center-stable manifolds is transverse at $q_1(0)$ and $\mathcal{W}_1$ is two-dimensional. By the implicit function theorem smoothness of stable and unstable manifolds implies that a neighborhood of $q_1(0)$ is a two-dimensional C^k manifold. Since $\mathcal{W}_1$ is flow invariant, the tangent space $T_{q_1(0)}\mathcal{W}_1$ is spanned by E_1 and $\dot{q}_1(0)$, which is transverse

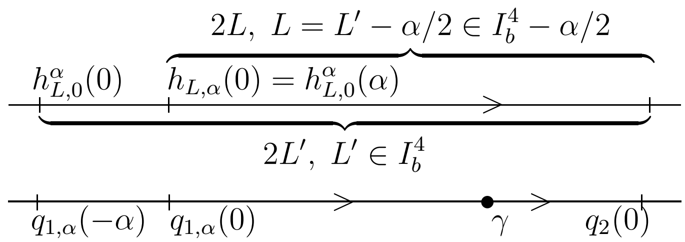

Fig. 9. Schematic picture of shifting the approximate semi-travel-time by $\alpha/2$.

to the strong stable fibers. This allows to find a C^k curve Γ through $q_1(0)$, which is simultaneously transverse to the flow and the strong stable fibers. Since $\mathcal{W}^{\mathrm{s}}(\gamma)$ is fibered by $\mathcal{W}^{\mathrm{ss}}_{\alpha}(\gamma)$ we can parametrize any such Γ as claimed. □

Next, we identify the sets of L associated to $q_{1,\alpha}$ (not $q_{1,\alpha}(\cdot-\alpha)$) and those associated to Γ as $I_\alpha := I_b^4 - \alpha/2$ and $I_\Gamma := \cup_{\alpha\in J} I_\alpha \subset \mathbb{R}$ respectively.

Proposition 2 *Assume hypotheses 1, 2 and 4. Let $\Gamma = \Gamma_J$ as in lemma 8. There exist constants $\epsilon_\Gamma > 0$, $\delta_\Gamma > 0$ and j_0 and countably infinite C^k families of curves $\mu_\Gamma^j(L)$ of parameters and $h_{L,\Gamma}^j$ of 1–homoclinic orbits to p_0 for $L \in I_\Gamma$ and $j \geq j_0$ such that for each j the estimates of theorem 2 hold.*

There is $J' \subset J$ such that for any 1–homoclinic h which lies in the ϵ_Γ–neighborhood of $\mathcal{O}(\Gamma_{J'}) \cup \mathcal{O}(\{q_2(0)\})$ there exist $j \geq j_0$, $L \in I_\Gamma$ and $\sigma = O(\epsilon_\Gamma)$ such that $h(\cdot - \sigma) \equiv h_{L,\Gamma}^j$ and for $\alpha \in J'$ we have $\{h_{L,\mathfrak{v}}^{\alpha}(\xi) \mid \xi \in \mathbb{R}, |\mathfrak{v}| \leq \delta_\Gamma, L \in I_\alpha\} \subset \{h_{L,\Gamma}^j(\xi) | \xi \in \mathbb{R}, L \in I_\Gamma, j \geq j_0\}$.

Proof. We will us the notation as introduced above. The existence of $h_{L,\mathfrak{v}}^{\alpha}$ for parameters $\mu(L,\alpha,\mathfrak{v})$ follows from theorem 2, and $h_{L,0}^{\alpha}$ for parameter $\mu(L,\alpha,0)$ are defined for $L \in I_b^4$. As shown in figure 9, $h_{L,\alpha}(0) - q_{1,\alpha}(0) = O(\mathrm{e}^{(\eta-\kappa)L})$ with any $L \in I_\alpha$ and parameters $\mu(L,\alpha)$. Recall I_b has been derived in lemma 3 by slightly 'thickening' the sequence $T_j = jT_\gamma/2$, which has the property $|q_{1,\alpha}(T_j - \alpha) - q_2(-T_j)| \leq C\mathrm{e}^{-\kappa T_j}$. By definition of I_α we conclude for $\alpha, \alpha' \in I_\Gamma$ that $I_\alpha = I_{\alpha'} + (\alpha' - \alpha)/2$, and so the union for $\alpha \in J$ yields $L \in I_\Gamma$. Smoothness of $h_{L,\alpha}$ in α follows from that of Γ, the heteroclinic and the trichotomies with respect to parameters, cf. e.g. [30].

For each $q_{1,\alpha}(\cdot - \alpha)$ theorem 2 provides a constant $\epsilon(\alpha) = \epsilon_4$. Continuous dependence and boundedness of J yield uniform $0 < \epsilon := \min\{\epsilon(\alpha) \,|\, \alpha \in J\}$. Each interval in I_α is open, hence there are $\rho_j > 0$ such that for $\alpha, \alpha' \in J$, $|\alpha - \alpha'| \leq \rho_j$ we have $T_j \in I_\alpha \cap I_{\alpha'}$ and the intersection is open. Note that $I_\alpha \cap I_{\alpha'}$ is bounded, because the intervals constituting I_β are exponentially short for any β. Let I_L be the connected component of I_α containing L. By continuity of $h_{L,\beta}$ in β we may decrease the $\rho_j > 0$ such that for any $L \in I_\alpha \cap I_{\alpha'}$

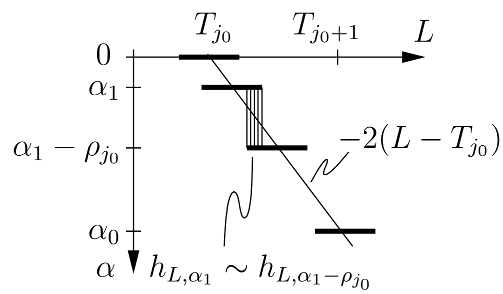

Fig. 10. Patching curves of 1–homoclinic orbits in the (α, L)-plane: in the shaded region orbits with same L coincide.

and $\beta = \alpha$, $\beta = \alpha'$ it holds that

$$(h_{L,\beta}(\cdot - \beta) - q_{1,\alpha}(\cdot - \alpha), h_{L,\beta}(2L - \beta + \cdot) - q_2) \in B^0_\epsilon(0; \sup I_L). \qquad (4.1)$$

Theorem 2 provides a unique $\sigma_L(\alpha) = O(\alpha - \alpha')$, C^k in L and α such that the overlapping parts can be patched together smoothly (note $\ell_L \equiv L$, because $L \in I_\alpha$), i.e. for a small $\mathfrak{v}$

$$\begin{aligned} h_{L,\alpha} &\equiv h^{\alpha'}_{L,\mathfrak{v}}(\cdot + \alpha - \alpha' + \sigma_L(\alpha)) \\ \mu(L, \alpha) &= \mu(L, \alpha', \tilde{Q} P^{\mathrm{u}}_{-1}(0)(h^{\alpha}_{L,\mathfrak{v}'}(\sigma_L - \alpha) - q_{1,\alpha}(-\alpha))). \end{aligned} \qquad (4.2)$$

Therefore, we can extend the curves of parameters and 1–homoclinic orbits, cf. figure 10 as follows. Let $\alpha_0 := \inf J$, $\alpha_1 := \sup J$, $j_0 := \min\{j \,|\, T_j \in I_\Gamma\}$ and for $j \geq j_0$, $L \in [T_j - \alpha_1/2, T_j - \alpha_0/2]$ set $\alpha(j, L) := -2(L - T_j)$. Due to (4.2) we can define C^k curves for $j \geq j_0$ and $L \in I_\Gamma$ by

$$h^j_{L,\Gamma} \;:=\; h_{L,\alpha(L,j)}\,, \quad \mu^j_\Gamma(L) \;:=\; \mu(L, \alpha(L, j)).$$

These contain all 1–homoclinic orbits $h_{L,\alpha}$ and parameters $\mu(L, \alpha)$, because by definition

$$\bigcup_{L \in I_\Gamma} \alpha(L, j) = I_\Gamma \text{ for each } j \geq j_0.$$

We emphasize that for $\alpha \neq \alpha'$ and $\alpha - \alpha' < T_\gamma$ we have $q_{1,\alpha}(0) \neq q_{1,\alpha'}(0)$, and $|h_{L,\alpha}(0) - q_{1,\alpha}(0)| \leq \mathrm{e}^{(\eta-\kappa)L}$. Hence, for L sufficiently large $h_{L,\alpha'}(\cdot + \tau) \neq h_{L,\alpha}$ for any τ, i.e. we can separate 1–homoclinic orbits by fixing α, α' and choosing L large, while (4.2) holds for α and α' close to each other.

As to the claimed local uniqueness, let $E^{\mathrm{u}}_{-1}(0; \alpha)$, $P^{\mathrm{u}}_{-1}(0; \alpha)$ and $E^{\mathrm{s}}_{+1}(0; \alpha)$, $P^{\mathrm{s}}_{+1}(0; \alpha)$ as well as $E_1(\alpha)$, $\tilde{Q}_\alpha$ denote the spaces and projections from theorem 2 with respect to the heteroclinic $q_{1,\alpha}(\cdot - \alpha)$. Given a 1–homoclinic h with variations $W(\sigma, L) \in B^0_\epsilon(0; L)$, the local uniqueness statement of theorem 2 implies that from $\tilde{Q}_\alpha P^{\mathrm{u}}_{-1}(0; \alpha) w_1(0; \sigma) = 0 \in E_1(\alpha)$ it follows h is a time shift of $h^\alpha_{L,\mathfrak{v}}$. Let $\beta \in J$, and set $q := q_{1,\beta}(0)$. The tangent space is $T_q \mathcal{W}_1 = \mathrm{span}\{E_1(\beta), \dot{q}_{1,\beta}(0))\}$, and by choice of Γ it follows $T_q \mathcal{W}_1 = \mathrm{span}\{\frac{\mathrm{d}}{\mathrm{d}\alpha}|_{\alpha=\beta} q_{1,\alpha}(\alpha), \dot{q}_{1,\beta}(0)\}$. Therefore, there is $\delta_\Gamma > 0$ such that for $|\mathfrak{v}| < \delta_\Gamma$

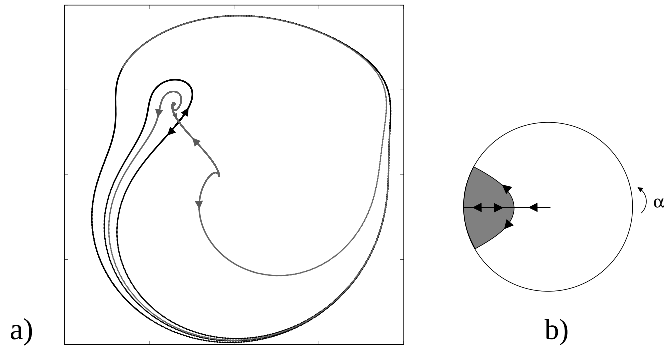

Fig. 11. a) Example for the case of winding number zero by an additional saddle: $x' = x - x(x^2+y^2) - y + 0.7\exp(-10(y-0.5)^2)$, $y' = y - y(x^2+y^2) + x$. The saddle's unstable manifold (black) separates heteroclinic sets from the central and the outer focus; computed with dstool [2]. b) schematic picture for 'cutting out' strong stable fibers with a saddle's unstable manifold.

we can solve $\tilde{Q}_\alpha P^{\mathrm{u}}_{-1}(0;\alpha)w_1(0;\sigma) = 0$ by the implicit function theorem and in addition $\{h^j_{L,\Gamma_J}(\xi) \mid \xi \in \mathbb{R}, L \in I_{\Gamma_J}, j \geq j_0\} \supset \{h^{\alpha_0}_{L,\mathfrak{v}}(\xi) \mid |\mathfrak{v}| \leq \delta_\Gamma, \xi \in \mathbb{R}\}$. By continuity, there are $J' \subset J$ and $\epsilon_\Gamma \in (0,\epsilon)$, such that for $\Gamma := \Gamma_{J'}$ we have

$$B_{\epsilon_\Gamma}(\{h^j_{L,\Gamma'}(\xi) \mid \xi \in \mathbb{R}, L \in I_{\Gamma'}, j \geq j_0\}) \subset \{h_{L,\Gamma}(\xi) \mid \xi \in \mathbb{R}, L \in I_\Gamma\}.$$

Together with the local uniqueness statement in theorem 2 we conclude the claimed uniqueness for any 1–homoclinic with variations in $B^0_{\epsilon_\Gamma}(0;L)$ with respect to $q_{1,\alpha}$, $\alpha \in \Gamma_{J'}$ and any $L \in I_{\Gamma_{J'}}$. □

If Γ_J is so that $I_\alpha \cap I_{\alpha'} \neq \emptyset$ implies $\alpha - \alpha' < T_\gamma$, then $\alpha \in J_\Gamma$ uniquely identifies $h_{L,\alpha}$. However, if $J = (\alpha_0, \alpha_1)$ and $|\alpha_0 - \alpha_1| \geq T_\gamma$, then the 1–homoclinic orbits h_{1,α_0} and h_{1,α_1} may or may not coincide. Moreover, $I_\Gamma = (L_*, \infty)$ for some L_* does not necessarily imply a connected curve of 1–homoclinic orbits (and parameters). However, we will show that the following hypothesis is a sufficient condition for the bifurcation of 1–homoclinic orbits.

Hypothesis 5 *The intersection $\mathcal{W}_1$ between the equilibrium's unstable manifold $\mathcal{W}^{\mathrm{u}}(p_0)$ and the periodic orbit's center-stable manifold $\mathcal{W}^{\mathrm{sc}}(\gamma)$ contains a nontrivial C^k Jordan curve Γ, which transversely intersects all strong stable fibers $\mathcal{W}^{\mathrm{ss}}_\alpha(\gamma)$, $0 \leq \alpha < T_\gamma$. Any heteroclinic orbit in $\mathcal{W}_1$ intersects Γ and those starting in Γ satisfy hypotheses 2 and 4.*

Here a Jordan curve is a non self-intersecting, closed curve. Roughly speaking, the hypothesis means that $\mathcal{W}_1$ has a nice parametrization. It is for instance satisfied when γ stems from a 'nearby' Hopf-bifurcation of p_0. In this case the union of p_0 with its center manifold contains a disc which has γ as its boundary, cf. e.g. chapters 6.4 and 8 in [15]. Circles in this disc which wind around p_0 once can be chosen for Γ, and for counter-examples to hypothesis

5 we can insert a saddle-node bifurcation into this disc, see figure 11.

Let Γ be a curve that satisfies hypothesis 5. As shown in proposition 2, transversality to strong stable fibers yields a parametrization $\Gamma = \{q_{1,\alpha}(0) \in \mathcal{W}^{\mathrm{ss}}_\alpha(\gamma) \,|\, \alpha \in [0,T^*)\}$. Since Γ is closed and nontrivial there is $m \in \mathbb{N} \setminus \{0\}$ such that $T^* = mT_\gamma$ and we call m the winding number of Γ. By surjectivity the flow provides a pointwise defined diffeomorphism between any two curves that satisfy hypothesis 5, so m counts how often the set of heteroclinic points $\mathcal{W}_1$ 'winds around' γ and p_0. If $\mathcal{W}_1 \cup \mathcal{O}(\gamma(0))$ is a Möbius band near γ, then $m = 2$.

Theorem 3 *Assume hypotheses 1 and 5, and let m be the winding number of Γ. There are positive constants ϵ, L_* and C, such that the following holds. There exist m curves h^j_L of 1–homoclinic orbits to p_0 of (2.1) and $\mu^j(L)$ of associated parameters for $L \geq L_4$. The curves are of class C^k, bifurcate from the heteroclinic cycle and satisfy*

$$
\begin{aligned}
|\mu^j(L)| &\leq C\mathrm{e}^{-2\kappa L}\\
||h_L - q_1||_{0,\tilde{L}} + ||h^j_L(2L - \cdot) - q_2||_{-0,\tilde{L}} &\leq C\mathrm{e}^{(\eta-\kappa)\tilde{L}}.
\end{aligned}
$$

Let h be a 1–homoclinic solution to (2.1) for $|\mu| \leq \delta$ such that $h(\xi) \in B_\epsilon(\mathcal{W}_1 \cup \{q_2(\xi)|\xi \in \mathbb{R}\}) \subset \mathbb{R}^n$ for all ξ. There exist a unique $\sigma \in \mathbb{R}$, $L \geq L_4$ and $j \in \{0, \ldots, m-1\}$ such that $h \equiv h^j_L(\cdot + \sigma)$ and $\mu = \mu^j(L)$.

Proof. Let Γ satisfy hypothesis 5 with winding number m. Parametrize $\Gamma = \{q_{1,\alpha}(0) \in \mathcal{W}^{\mathrm{ss}}_\alpha(\gamma) \,|\, \alpha \in [0,T^*]\}$ as in proposition 2 so that $q_{1,0} \equiv q_{1,T^*}$, i.e. $T^* = mT_\gamma$. Using the estimates and notation from theorem 2 as well as (4.1) for $L \in I_\Gamma$ it holds that

$$
\begin{aligned}
&||(h_{L,-T^*}(\cdot - T^*) - q_{1,-T^*}(\cdot - T^*), h_{L,-T^*}(2L - T^* + \cdot) - q_2)||_L \leq C\mathrm{e}^{(\eta-\kappa)L}\\
&\quad\Rightarrow ||(h_{L,-T^*} - q_{1,-T^*}, h_{L,-T^*}(2L - T^* + \cdot) - q_2)||_L \leq C\mathrm{e}^{\eta T^*}\mathrm{e}^{(\eta-\kappa)L}\\
&\quad\Leftrightarrow ||(h_{L,-T^*} - q_{1,0}, h_{L,-T^*}(2(L - T^*/2) + \cdot) - q_2)||_L \leq C\mathrm{e}^{\eta T^*}\mathrm{e}^{(\eta-\kappa)L}.
\end{aligned}
$$

For sufficiently large $\bar{L}_4$ and with ϵ_Γ from proposition 2 we have $C\mathrm{e}^{T^*}\mathrm{e}^{(\eta-\kappa)L} \leq \epsilon_\Gamma$ for any $L \geq \bar{L}_4$. Hence, by uniqueness $h_{L+T^*/2,0} \equiv h_{L,-T^*}$, where $\ell_L = L$ and $\sigma_L = 0$, because both vary with respect to the same heteroclinic orbit for fixed points of $\mathcal{G}$, cf. lemma 4. Therefore, the curves found in proposition 2 consist of m curves parametrized by $j = 0, \ldots, m-1$ and $\alpha(L) := -2L \bmod T^*$ as

$$
h^j_L \;:=\; h_{(L+jT_\gamma/2),\alpha(L)}\;, \quad \mu^j(L) \;:=\; \mu(L + jT_\gamma/2, \alpha(L)). \tag{4.3}
$$

We next show that h^j_L is not a time shift of any $h^{j'}_{L'}$ if $j \neq j'$ and L, L' are large enough, and that any 1–homoclinic near the heteroclinic cycle is

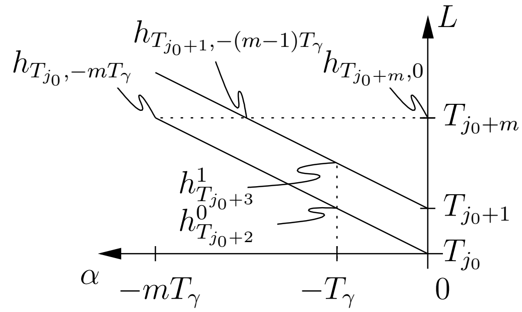

Fig. 12. Paths of the 1–homoclinic orbits in the parameters α and L using the sequence $T_j := jT_\gamma/2$ and j_0 so that $T_{j_0} \in I_b^4$. The homoclinic orbits for $L = (m + j_0)T_\gamma/2$ and $\alpha = 0$, $\alpha = -mT_\gamma$ coincide, e.g. $h_{T_{j_0},-mT_\gamma} \equiv h_{T_{j_0+m},0}$.

captured. It follows that there are precisely m curves of 1–homoclinic orbits bifurcating from the heteroclinic cycle, as indicated in figure 12, and precisely m connected associated parameter curves $\mu^j(L)$ bifurcate from $\mu = 0$.

Since 1–homoclinic orbits in a uniform neighborhood of $q_{1,\alpha}$ and q_2 for each α are locally unique by theorem 2, it suffices to show that none of the heteroclinic orbits $q_{1,\alpha}$ are related by time shifts. Assume for a contradiction that there are $\xi_0 > 0$ and $\alpha_0, \alpha_1 \in I_\Gamma$ such that $0 < \alpha_0 - \alpha_1 < mT_\gamma$ and $q_{1,\alpha_0}(\xi_0) = q_{1,\alpha_1}(0)$. Define the partial orbit $O_1 := \{q_{1,\alpha_0}(\xi) \,|\, 0 \le \xi \le \xi_0\}$ and the closed curve $\Gamma_0 := O_1 \cup \{q_{1,\alpha}(0) \,|\, \alpha \in (\alpha_0, \alpha_1)\}$. We may assume $\xi_0 > 0$ is the ξ with smallest absolute value so that $q_{1,\alpha_0}(\xi) = q_{1,\alpha_1}(0)$. Since Γ is non self-intersecting Γ_0 is non self-intersecting. The curves Γ and Γ_0 are bounded, so we can find $\xi_+ > \xi_- > \xi_0$ such that the partial orbits $\{\Phi(\xi)\Gamma_0 \,|\, -\xi_+ \le \xi \le -\xi_-\}$ and $\{\Phi(\xi)\Gamma_0 \,|\, \xi_- \le \xi \ge \xi_+\}$ have empty intersection with Γ and Γ_0. Therefore, the partial orbit $O := \{\Phi(\xi)\Gamma_0 \,|\, -\xi_+ \le \xi \le \xi_+\} \subset W_1$ is a C^k manifold which is homeomorphic to an annulus, because $\Phi(\xi)\Gamma_0$ is homeomorphic to S^1 for any ξ. Hence, O is separated by Γ_0 into an 'interior' containing $\Phi(\xi_+)\Gamma$, and an 'exterior' containing $\Phi(-\xi_+)\Gamma$. Any connected curve in $\mathcal{W}_1$ that has parts inside and outside has to cross ∂O or Γ_0.

The parametrization induces an orientation on Γ for increasing α, and smoothness yields tangent vectors $\partial_\alpha q_{1,\alpha}(0) \neq 0$. Since Γ is transverse to the flow, the angle $s(\alpha)$ from $\partial_\alpha q_{1,\alpha}(0)$ to $f(q_{1,\alpha}(0); 0)$ in the two-dimensional tangent space $T_{q_{1,\alpha}(0)}\mathcal{W}_1$ is never a multiple of π. By assumption, this angle always lies in $(\pi, 2\pi)$, see figure 13. Therefore, for increasing α the curve Γ can cross the tangent vectors to the flow line O_1 only at angles $s(\alpha) \in (\pi, 2\pi)$, i.e. from outside Γ_0 to inside. Hence, for any sufficiently small $\beta > 0$ we have $q_{1,\alpha_1+\beta}(0)$ and $q_{1,\alpha_0-\beta}(0)$ lie on opposite sides of Γ_0 in O, see figure 13. Since Γ is closed and does not intersect ∂O, it has to cross Γ_0 from interior to exterior for increasing α. This would have to occur at O_1, because Γ is non self-intersecting, which contradicts $s(\alpha) \in (\pi, 2\pi)$. Hence such an intersection along a flow line cannot occur for $\xi_0 > 0$ and $0 < \alpha_0 < \alpha_1 < mT_\gamma$. Similarly, the case $\xi_0 < 0$ is ruled out, so the set of heteroclinic orbits is distinct. As noted above, it

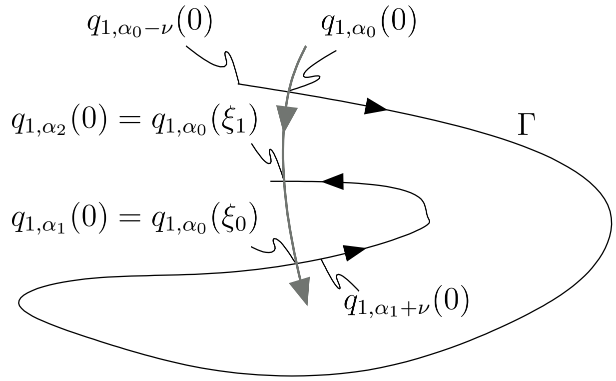

Fig. 13. Configuration in the case that $\mathcal{O}(\Gamma)$ intersects Γ along a flow line (gray).

follows that the 1–homoclinic orbits h_L^j are distinct.

The constants C, L_4 and δ_4 from theorem 2 may be chosen uniform in α due to continuity and the boundedness of $[0, T_\gamma]$. We denote by L_* the uniform L_4 and set $\epsilon := \epsilon_\Gamma$ from proposition 2.

Let u be a solution to (2.1) with $u(0) \in B_\epsilon(\Gamma) \subset \mathbb{R}^n$ and L so that $u(2L) \in B_\epsilon(q_2(0))$. Then $L \to \infty$ as $\epsilon \to 0$. Since $\mathcal{O}(\Gamma \cup q_2(0))$ is bounded and the vector field continuous, there is a uniform lower bound L_ϵ such that for all solutions u and times L with $u(0)$ and $u(2L)$ as above we have $L \geq L_\epsilon$, and $L_\epsilon \to \infty$ as $\epsilon \to 0$. Now let h be a 1–homoclinic orbit that lies in $B_\epsilon(\mathcal{W}_1 \cup \{q_2(\xi) | \xi \in \mathbb{R}\}) \subset \mathbb{R}^n$. There is L and, by hypothesis 5, there are α and σ such that $|h(\sigma) - q_{1,\alpha}(0)| \leq \epsilon$ and $|h(2L + \sigma) - q_2(0)| \leq \epsilon$. Therefore $L \geq L_\epsilon$ and we may assume ϵ is so small that $L_\epsilon \geq L_4$ and $\epsilon \leq \epsilon_\Gamma$, where $\epsilon_\Gamma > 0$ is from proposition 2. It follows from proposition 2 for unique σ_L and $j \in \{0, \dots, m-1\}$ that $h \equiv h_L^j(\cdot + \sigma_L)$. $\square$

5 Expansion of parameter curves

In this section, we investigate the leading order geometry of any connected parameter curve $\mu(L)$ from theorem 3. Let $\Gamma = \{q_{1,\alpha}(0) \in \mathcal{W}_\alpha^{\mathrm{ss}}(\gamma) \,|\, \alpha \in [0, mT_\gamma)\}$ be a curve which satisfies hypothesis 5 and has winding number m. Theorem 3 and (4.3) imply that initial conditions for the heteroclinic orbits that are exponentially close to the 1–homoclinic orbits at parameter values $\mu(L)$ can be chosen $q_{1,\alpha(L)}(0)$ with $\alpha(L) = -2L \mod mT_\gamma$. Note that for these curves $\mathfrak{v} = 0$, but the associated projections P_L^{s}, evolution $\Phi_1(\xi, \zeta)$ and variations $W^4(L)$, $W_0^4(L)$ indirectly depend on α.

By theorem 3 we have $\mu(L) = O(\mathrm{e}^{-2\kappa L})$ and so (3.40) yields

$$\mathcal{M}\mu(L) = -\langle P_L^{\mathrm{s}} b_L, a^1(-L)\rangle a_0^1 - \langle P_L^{\mathrm{s}} b_L, a^2(-L)\rangle a_0^2 + O(\mathrm{e}^{(\eta-3\kappa)L}). \qquad (5.1)$$

We expect that the parameter curve is determined to leading order as $L \to \infty$ by the scalar products in this expression. However, we will first refine the estimate of the remainder term to show that the unstable spectral gap of γ is irrelevant.

To obtain a refined leading order expansion we distinguish the leading stable rate $\kappa^{\mathrm{s}} \geq \kappa$ near the periodic orbit γ, i.e.

$$\kappa^{\mathrm{s}} := \min\{|\Re(\nu)| \,|\, \nu \in \mathrm{spec}(R),\, \Re(\nu) < 0\}$$

Lemma 9 *Under the assumptions of theorem 3, there exist $\delta > 0$ such that*

$$\mathcal{M}\mu(L) = -\langle P_L^{\mathrm{s}} b_L, a^1(-L)\rangle a_0^1 - \langle P_L^{\mathrm{s}} b_L, a^2(-L)\rangle a_0^2 + O(\mathrm{e}^{-(2\kappa^{\mathrm{s}}+\delta)L}).$$

Proof. The term of order $O(\mathrm{e}^{(\eta-3\kappa)L})$ in (5.1) stems from constant the term in (3.38) and enters (5.1) via the implicit function theorem using (3.41). Therefore, if the remainder term in (3.38) is in fact $O(\mathrm{e}^{2\kappa^{\mathrm{s}}+\delta})$, then the remainder term in (5.1) is of this order.

Since $\Phi^{\mathrm{s}}_{-2}(0,-L) = O(\mathrm{e}^{(\epsilon-\kappa^{\mathrm{s}})L})\ \forall\epsilon > 0$ it suffices to show that there is $\delta > 0$ with

$$|P_L^{\mathrm{s}}\left(\mathfrak{c}_1(W_0^4(L), L) + \mathfrak{c}_2(W^4(L), L, \mu(L))\right)| = O(\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta)L}).$$

We first consider $P_L^{\mathrm{s}}\mathfrak{c}_1(W_0^4(L), L)$. By definition $\mathfrak{c}_1(W_0, L) = \Phi^{\mathrm{u}}_{-2}(-L,0)\tilde{w}_2 - \Phi^{\mathrm{s}}_{+1}(L,0)\tilde{w}_1$ and by theorem 3 there is $\delta > 0$ such that $W_0^4(L) = O(\mathrm{e}^{-\delta L})$. On the one hand, from (2.7) we have $\Phi^{\mathrm{s}}_{+1}(L,0) = O(\mathrm{e}^{-\kappa^{\mathrm{s}}L})$. On the other hand, for possibly shrunken $\delta > 0$ we have $P_L^{\mathrm{s}}\Phi^{\mathrm{u}}_{-2}(-L,0) = O(\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta)L})$, because of the following. If γ is an equilibrium, it follows from lemma 1.1 ii) in [30] that

$$P_L^{\mathrm{s}} - P^{\mathrm{s}}_{-2}(-L) = O(\mathrm{e}^{(\epsilon-\kappa^{\mathrm{s}})L}) \qquad (5.2)$$

for any sufficiently small $\epsilon > 0$. The proof can be modified for our purposes by applying it to appropriately shifted variational equation $\dot{v} = (A_2(\xi) - \tilde{\eta})v$, similar to the proof of lemma 1. Since $\Phi^{\mathrm{u}}_{-2}(-L,0) = P^{\mathrm{u}}_{-2}(-L)\Phi_2(-L,0) = O(\mathrm{e}^{-\kappa L})$ the claim follows by choosing $\epsilon + \delta \leq \kappa$.

We next consider $P_L^{\mathrm{s}}\mathfrak{c}_2(W^4(L), L, \mu(L))$. By definition we have

$$\begin{aligned}\mathfrak{c}_2(W; L, \mu(L)) := &\int_0^{-L} \Phi^{\mathrm{u}}_{-2}(-L,\zeta) g_2(w_2(\zeta),\zeta;\mu)\mathrm{d}\zeta \\ &- \int_0^{L} \Phi^{\mathrm{s}}_{+1}(L,\zeta) g_1(w_1(\zeta),\zeta;\mu)\mathrm{d}\zeta.\end{aligned}$$

Again using $P^{\mathrm{s}}_L\Phi^{\mathrm{u}}_{-2}(-L,\zeta) = O(\mathrm{e}^{(\epsilon-\kappa^{\mathrm{s}})L}|\Phi^{\mathrm{s}}_{-2}(-L,\zeta)|)\ \forall\zeta \geq -L$ and $W^4(L) = O(\mathrm{e}^{(\eta-\kappa)L})$ as well as (2.6) we find a $\delta > 0$ such that in addition it holds that

$$P^{\mathrm{s}}_L \int_0^{-L} \Phi^{\mathrm{u}}_{-2}(-L,\zeta) g_2(w_2^4(L)(\zeta),\zeta;\mu(L))\mathrm{d}\zeta = O(\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta)L}).$$

For the claimed estimate it remains to show that

$$P^{\mathrm{s}}_L \int_0^{L} \Phi^{\mathrm{s}}_{+1}(L,\zeta) g_1(w_1^4(L)(\zeta),\zeta;\mu(L))\mathrm{d}\zeta = O(\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta)L}).$$

We obtain this estimate in suitable coordinates and weighted spaces. Firstly, we change coordinates so that $\mathcal{W}^{\mathrm{sc}}(\gamma) = \cup_{\alpha\in[0,T_\gamma]} E^{\mathrm{sc}}_\gamma(\alpha)$ near γ as in Lemma 3.6 in [30]. In these coordinates the nonlinearity satisfies $|P^{\mathrm{s}}_{+1}(\xi)g_1(w_1,\xi,\mu)| \leq C(\mathrm{e}^{(\epsilon-\kappa^{\mathrm{s}})L}|\mu| + (|w_1^{\mathrm{s}}| + \mathrm{e}^{(\epsilon-\kappa^{\mathrm{s}})\xi}|w_1| + |\mu|)|w_1|\ \forall\epsilon > 0$, because of the following. In [30], lemma 3.13, this estimate is shown for the projection of g with $P^{\mathrm{s}}_\gamma(\xi)$ and the claim follows from (5.2).

Secondly, we choose any $\epsilon \in (0,\kappa)$ and replace the space $\mathcal{X}_L$ in section 3.1 by the space $\mathcal{X}_L^{\kappa^{\mathrm{s}}-\epsilon} := \mathcal{C}_{(\kappa^{\mathrm{s}}-\epsilon),\tilde{L}} \times \mathcal{C}_{\eta,\tilde{L}} \times \mathcal{C}_{-\eta,\tilde{L}}$ with the weighted spaces as defined in section 3.1 for any $\eta \in (0,\kappa-\epsilon)$. We decompose $w_1 = w_1^{\mathrm{s}} + w_1^{\mathrm{cu}}$ using the trichotomy and take $W = (w_1^{\mathrm{s}}, w_1^{\mathrm{cu}}, w_2)$ in this space. The estimates of the nonlinearity used in section 3 hold in these coordinates. Lemma 5, which contains the crucial uniform boundedness of the linear operator $\mathcal{I}(L)$, follows in the same way by the choice of weight. Therefore, the results of section 3 and 4 remain valid and we can estimate $|w_1^{\mathrm{s}}(\xi)| \leq C\mathrm{e}^{(\epsilon-\kappa^{\mathrm{s}})\xi}$, hence

$$\begin{aligned}
&|\int_0^L \Phi^{\mathrm{s}}_{+1}(L,\zeta) g_1(w_1^4(L)(\zeta),\zeta;\mu(L))\mathrm{d}\zeta| \\
&\leq C\int_0^L \mathrm{e}^{(\epsilon-\kappa)^{\mathrm{s}}(L-\zeta)}\left(|(w^4(L))_1^{\mathrm{s}}(\zeta)| + \mathrm{e}^{(\epsilon-\kappa^{\mathrm{s}})\zeta}|(w^4(L))_1(\zeta)|\right)|(w^4(L))_1(\zeta)|\mathrm{d}\zeta \\
&\leq C\int_0^L \mathrm{e}^{(\epsilon-\kappa)^{\mathrm{s}}L}\cdot\mathrm{e}^{\epsilon\zeta}\cdot\mathrm{e}^{-\eta\zeta+(\eta-\kappa)L}\mathrm{d}\zeta \leq C\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta)L},
\end{aligned}$$

where the estimate from theorem 3 was used and $\delta \in (0,\kappa-\eta-\epsilon)$. □

As to the scalar products in (5.1), $b_L = q_2(-L) - q_{1,\alpha(L)}(L)$ by definition, so

$$\begin{aligned}
\langle -P^{\mathrm{s}}_L b_L, a^j(-L)\rangle = &\langle P^{\mathrm{s}}_L(q_{1,\alpha(L)}(L) - \gamma(-L)), a^j(-L)\rangle - \\
&-\langle P^{\mathrm{s}}_L(q_2(-L) - \gamma(-L)), a^j(-L)\rangle.
\end{aligned} \tag{5.3}$$

Recall the Floquet representation $A_{\mathrm{per}}(\xi)\mathrm{e}^{R\xi}$ of the evolution $\Phi_\gamma(\xi,0)$ of $v' = \partial_u f(\gamma(\xi))v$, where $A_{\mathrm{per}}(\xi)$ is T_γ-periodic and invertible, $A_{\mathrm{per}}(0) = \mathrm{Id}$. By hypothesis 1, the matrix R has an algebraically simple eigenvalue zero, and no other eigenvalues lie on the imaginary axis. Let E_j be the generalized eigenspace of R to its eigenvalues ν_j, $j = 1,\dots n$, and $\nu^{\rm s/u} \neq 0$ be the leading eigenvalues, i.e. $\Re(\nu^{\rm s}) < 0$ and $\Re(\nu_j) < 0 \Rightarrow \Re(\nu_j) \leq \Re(\nu^{\rm s})$ as well as $\Re(\nu^{\rm u}) > 0$ and $\Re(\nu_j) > 0 \Rightarrow \Re(\nu_j) \geq \Re(\nu^{\rm s})$. Then $\kappa^{\rm s/u} = |\Re(\nu^{\rm s/u})|$, if $\nu^{\rm s/u}$ is algebraically and geometrically simple. In addition, denote $\delta^{\rm s} := \min\{\nu_j - \nu^{\rm s} \,|\, \Re(\nu_j) < 0, \nu_j \neq \nu^{\rm s}\}$ as well as $\delta^{\rm u} := \min\{\nu_j - \nu^{\rm u} \,|\, \Re(\nu_j) < 0, \nu_j \neq \nu^{\rm u}\}$, where an empty set has minimum $-\infty$. We define the strong stable, leading stable and center-unstable generalized eigenspaces of R:

$$E_R^{\rm ss} := \sum_{\{j\,|\,\Re(\nu_j)<\Re(\nu^{\rm s})\}} E_j\ , \quad E_R^{\rm s} := \sum_{\{j\,|\,\Re(\nu_j)=\Re(\nu^{\rm s})\}} E_j\ , \quad E_R^{\rm cu} := \sum_{\{j\,|\,\Re(\nu_j)>\Re(\nu^{\rm s})\}} E_j.$$

Notice that $E_R^{\rm ss} \oplus E_R^{\rm s} \oplus E_R^{\rm cu} = \mathbb{R}^n$ and let $P_R^{\rm ss}$ be the projection onto $E_R^{\rm ss}$ with kernel $E_R^{\rm s} \oplus E_R^{\rm cu}$, and $P_R^{\rm s}$ the projection onto $E_R^{\rm s}$ with kernel $E_R^{\rm ss} \oplus E_R^{\rm cu}$, as well as $P_R^{\rm cu}$ the projection onto $E^{\rm cu}$ with kernel $E_R^{\rm ss} \oplus E^{\rm s}$.

Let $\mathcal{W}_\rho \subset \mathcal{W}_1$ be such that for $q(0) \in \mathcal{W}_\rho$ the solution $q(\xi)$ satisfies

$$\limsup_{\xi\to\infty} \frac{\ln(q(\xi))}{-\xi} = \rho.$$

Lemma 10 *Assume hypotheses 1 and 5 with winding number m, and hypothesis 6 or 7. There are coordinates such that $P_L^{\rm s}(q_2(-L)-\gamma(-L)) = O(\mathrm{e}^{-(\kappa^{\rm s}+\kappa)L})$ for $L \geq L_*$ and there exists $j \geq 1$ and a (mT_γ/j)–periodic C^k function $v : \mathbb{R} \to \Re(E_R^{\rm s})$ such that for $\xi \geq 0$*

$$P_L^{\rm s}(q_{1,\alpha}(-L) - \gamma(-L)) = A_{\rm per}(-L)\mathrm{e}^{RL}v(-2L) + O(\mathrm{e}^{-(\kappa^{\rm s}+\delta)L}) \tag{5.4}$$

for any $\delta < \min\{\kappa^{\rm s}, \delta^{\rm s}\}$. If $\mathcal{W}_{\kappa^{\rm s}} \neq \emptyset$ then there is $\alpha \in [0, mT_\gamma)$ such that $v(\alpha) \neq 0$.

Proof. We first show the expansion for $q_{1,\alpha}(\xi)$. Consider $v = q_{1,\alpha} - \gamma(\cdot + \alpha)$, which solves the variational equation

$$\begin{aligned} v' &= \partial_u f(\gamma(\xi+\alpha);0)v + g(v,\xi,\alpha) \quad \text{where} \\ g(v,\xi,\alpha) &= f(v+\gamma(\xi+\alpha);0) - f(\gamma(\xi+\alpha)) - \partial_u f(\gamma(\xi+\alpha);0)v. \end{aligned}$$

Analogous to (2.5) and (2.6) we obtain $g(v,\xi,\alpha) = O(|v|^2)$, because of the periodicity in α. We next consider $w = A_{\rm per}(\xi+\alpha)^{-1}v$. It holds for general Floquet representation, cf. e.g. [7], that $\partial_u f(\gamma(\xi);0)A_{\rm per}(\xi) = A_{\rm per}(\xi)' + A_{\rm per}(\xi)R$ and so w solves

$$w' = Rw + A_{\rm per}^{-1}(\xi+\alpha)g(A_{\rm per}(\xi+\alpha)w,\xi,\alpha).$$

Since $\nu^{\rm s}$ is simple, it follows $v(\xi) = O(\mathrm{e}^{-\kappa^{\rm s}\xi})$ for $\xi \geq 0$, and so $w(\xi) = O(\mathrm{e}^{-\kappa^{\rm s}\xi})$ by periodicity. Therefore, for any α, the proof of Theorem 4.5 from chapter 13 in [7] implies that there is a vector $v(\alpha) \in \Re(E_R^{\rm s}) \oplus \Im(E_R^{\rm s})$ such that $w(\xi) = \mathrm{e}^{R\xi}v(\alpha) + O(\mathrm{e}^{-(\kappa^{\rm s}+\delta)\xi})$ for $\xi \geq 0$ and any $\delta < \min\{\kappa^{\rm s}, \delta^{\rm s}\}$. Moreover,

$v(\alpha)$ satisfies

$$
\begin{aligned}
v(\alpha) = P^{\mathrm{s}}\big((P_R^{\mathrm{ss}} - \mathrm{Id})w(0) \\
+ \textstyle\int_0^\infty \mathrm{e}^{-R\zeta}(P_R^{\mathrm{s}} + P_R^{\mathrm{cu}})A_{\mathrm{per}}^{-1}(\zeta+\alpha)g(A_{\mathrm{per}}(\zeta+\alpha)w(\zeta),\zeta,\alpha)\mathrm{d}\zeta\big).
\end{aligned}
$$

Since $P_R^{\mathrm{ss}} + P_R^{\mathrm{cu}} + P_R^{\mathrm{s}} = \mathrm{Id}$ and $w(\zeta) = A_{\mathrm{per}}^{-1}(\alpha+\zeta)(q_{1,\alpha}(\zeta) - \gamma(\zeta+\alpha))$ we obtain

$$
\begin{aligned}
v(\alpha) = -P_R^{\mathrm{s}}A_{\mathrm{per}}^{-1}(\alpha)(q_{1,\alpha}(0) - \gamma(\alpha)) + \\
+ \textstyle\int_0^\infty \mathrm{e}^{-R\zeta}P_R^{\mathrm{s}}A_{\mathrm{per}}^{-1}(\zeta+\alpha)g(q_{1,\alpha}(\zeta) - \gamma(\zeta+\alpha),\zeta,\alpha)\mathrm{d}\zeta.
\end{aligned}
\tag{5.5}
$$

By periodicity in α the integral converges uniformly in α and so $v(\alpha)$ is of class C^k. Since $q_{1,\alpha}(0)$ has minimal period mT_γ uniqueness of the solutions imply that $q_{1,\alpha}(\xi)$ has minimal period mT_γ for any $\xi \in \mathbb{R}$. Hence, (5.5) implies that $v(\alpha)$ has period mT_γ/j for some $j \geq 1$. Changing coordinates back to u and substituting $\alpha = \alpha(L) = -2L \mod mT_\gamma$ we obtain

$$
q_{1,\alpha(L)}(L) = \gamma(-L) + A_{\mathrm{per}}(-L)\mathrm{e}^{RL}v(-2L) + O(\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta)L}).
$$

On the other hand $P_L^{\mathrm{s}} - P_\gamma^{\mathrm{s}}(-L) = O(\mathrm{e}^{-\kappa^{\mathrm{s}}L})$ and $A_{\mathrm{per}}(-L)\mathrm{e}^{RL}v(-2L) \in E_\gamma^{\mathrm{s}}(-L)$, hence the claimed expansion holds.

As to the expansion of $q_2(\xi)$ which lies in the unstable manifold, suitable coordinates which straighten the unstable fibers near γ, see lemma 3.10 in [30], yield $P_\gamma^{\mathrm{s}}(-L)(q_2(-L) - \gamma(-L)) = 0$. Since $q_2(-L) - \gamma(-L) = O(\mathrm{e}^{\kappa L})$ the claim follows from (5.2).

As to roots of $v(\alpha)$, if $\mathcal{W}_{\kappa^{\mathrm{s}}} \neq \emptyset$ then by hypothesis 5 there are $\sigma \in \mathbb{R}$ and $\alpha \in [0, mT_\gamma)$ such that $q(0) = q_{1,\alpha}(\sigma)$, and so (5.4) implies that $v(\alpha) \neq 0$. $\square$

Hence, while it is possible that $v(\alpha)$ is constant, the leading order term in (5.4) does not vanish identically if $\mathcal{W}_1$ contains a leading order strong stable fiber, i.e. $\mathcal{W}_{\kappa^{\mathrm{s}}} \neq \emptyset$.

Next we prove a leading order expansion as $L \to \infty$ of the parameter curve $\mu(L)$. In the following A^* denotes the adjoint of A and $E^\perp$ the ortho-complement of a linear space E with respect to the standard scalar product, $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$, and direct sums as well as spans are over $\mathbb{R}$. Let $\tilde{E}_\rho \subset (E_2)^\perp$ be the set where solutions $a(\xi)$ to the adjoint linear equation $a' = -(\partial_u f(q_2(\xi)))^* a$ with $a(0) \in \tilde{E}_\rho$ satisfy

$$
\limsup_{\xi \to -\infty} \frac{\ln(a(\xi))}{\xi} = \rho.
$$

Since the equation is linear it follows that $E_\rho := \tilde{E}_\rho \cup \{0\}$ is a linear subspace of $(E_2)^\perp$.

Hypothesis 6 (real leading eigenvalue)
The eigenvalue ν^{s} is real and algebraically and geometrically simple.

Hypothesis 7 (complex leading eigenvalues)
The eigenvalues $\nu^{\mathrm{s}} \neq \overline{\nu^{\mathrm{s}}}$ are algebraically and geometrically simple.

Theorem 4 *Assume hypotheses 1, 2, 4, 5. Let m be the winding number of Γ and $\mu(L)$ a parameter curve of 1–homoclinic orbits to p_0 from theorem 3.*

Assume hypothesis 6. If $\tilde{E}_{\kappa^{\mathrm{s}}} \neq \emptyset$, then there exist $b_0 \in \mathbb{R}^2$, $\delta > 0$ and a C^k function $s_0 : \mathbb{R} \to \mathbb{R}$, which is constant or has minimal period $(mT_\gamma/(2\ell))$ for some $\ell \in \mathbb{N} \setminus \{0\}$ and

$$\mu(L) = \mathrm{e}^{-2\kappa^{\mathrm{s}} L} s_0(L) b_0 + O(\mathrm{e}^{-(2\kappa^{\mathrm{s}}+\delta)L}).$$

Assume hypothesis 7. Then $\nu^{\mathrm{s}} = -\kappa^{\mathrm{s}} + \mathrm{i}\sigma$ for some $\sigma \in \mathbb{R} \setminus \{0\}$. If $\dim(E_{\kappa^{\mathrm{s}}}) = 2$, then there exist $\delta > 0$ and C^k functions $b_j : \mathbb{R} \to \mathbb{R}^2$, $j = 1, 2$, which are constant or have minimal period $(mT_\gamma/(2\ell))$ for some $\ell \in \mathbb{N} \setminus \{0\}$ and

$$\mu(L) = \mathrm{e}^{-2\kappa^{\mathrm{s}} L} \left(\sin(2L\sigma) b_1(L) + \cos(2L\sigma) b_2(L)\right) + O(\mathrm{e}^{-(2\kappa^{\mathrm{s}}+\delta)L}).$$

If $\dim(E_{\kappa^{\mathrm{s}}}) = 1$ then the same holds and there exist $\tilde{b}_0 \in \mathbb{R}^2$, $s_j : \mathbb{R} \to \mathbb{R}$ such that $b_j(L) = s_j(L)\tilde{b}_0$ for $j = 1, 2$.

Assume $\mathcal{W}_{\kappa^{\mathrm{s}}} \neq \emptyset$, $\tilde{E}_{\kappa^{\mathrm{s}}} \neq \emptyset$. There exists $\alpha \in [0, mT_\gamma)$ such that $q_{1,\alpha}(0) \in \mathcal{W}_{\kappa^{\mathrm{s}}}$ and $v(\alpha) \in (\Re(E^{\mathrm{s}}_{R^*}))^\perp$ is equivalent to $s_0(-\alpha/2) = 0$ or $b_j(-\alpha/2) = 0$, $j = 1, 2$.

Proof. We first derive expansions under hypothesis 6 and 7 alternatively. Notice that by simplicity of ν^{s} we have $\Re(\nu^{\mathrm{s}}) = -\kappa^{\mathrm{s}}$. Consider the scalar product in (5.3) involving $q_2(-L)$. Since $a^j(\xi) = O(\mathrm{e}^{-\kappa^{\mathrm{s}}|\xi|})$ for $\xi \leq 0$ lemma 10 implies

$$\langle (P^{\mathrm{s}}_L(q_2(-L) - \gamma(-L)), a^j(-L)\rangle = O(\mathrm{e}^{-(2\kappa^{\mathrm{s}}+\kappa)L}). \tag{5.6}$$

For the scalar product in (5.3) involving $q_{1,\alpha(L)}(-L)$ we consider the adjoint solution $a^j(\xi)$ in more detail. Notice that $(P^{\mathrm{s}}_R)^*$ is the projection onto the leading stable eigenspace of R^*, and denote $E^{\mathrm{s}}_{R^*} := \mathrm{Rg}((P^{\mathrm{s}}_R)^*)$. We can write

$$\dot{a}^j = -(\partial_u f(\gamma(\xi))^* + B(\xi)^*)a^j$$

where $B(\xi) := \partial_u f(q_2(\xi)) - \partial_u f(\gamma(\xi))$. Since ν^{s} is simple and $B(\xi) = O(\mathrm{e}^{-\kappa^{\mathrm{s}} L})$, problem 33 from chapter 3 in [7] applies with minor modifications. It follows that there is a constant vector $c(a^j) \in \Re(E^{\mathrm{s}}_{R^*})$ and $\delta > 0$ such that

$$a^j(\xi) = (\Phi_\gamma(0,\xi))^* c(a^j) + O(\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta)|\xi|}), \text{ as } \xi \to -\infty.$$

Using the Floquet representation $(\Phi_\gamma(0,\xi))^* = (A^*_{\mathrm{per}}(\xi))^{-1}\mathrm{e}^{-R\xi}$ for the adjoint evolution we obtain that for all $\epsilon > 0$ it holds that

$$a^j(-L) = \mathrm{e}^{R^*L}(A^{-1}_{\mathrm{per}}(-L))^* c(a^j) + O(\mathrm{e}^{-(\kappa^{\mathrm{s}}+\delta^{\mathrm{s}}-\epsilon)L}). \tag{5.7}$$

Application of (5.6), lemma 10 and (5.7) to (5.3) implies that there exists $\delta > 0$ such that

$$\begin{aligned}
&\langle -P^{\mathrm{s}}_L b_L, a^j(-L)\rangle \\
&= \langle A_{\mathrm{per}}(-L)\mathrm{e}^{RL}v(-2L), (A_{\mathrm{per}}(-L)^*)^{-1}\mathrm{e}^{R^*L}c(a^j)\rangle + O(\mathrm{e}^{-(2\kappa^{\mathrm{s}}+\delta)L}) \qquad (5.8)\\
&= \langle v(-2L), \mathrm{e}^{2R^*L}c(a^j)\rangle + O(\mathrm{e}^{-(2\kappa^{\mathrm{s}}+\delta)L}).
\end{aligned}$$

Upon substituting (5.8) into (5.1) lemma 9 implies that there is $\delta > 0$ so

$$\mu(L) = \sum_{j=1,2} \langle v(-2L), \mathrm{e}^{2R^*L}c(a^j)\rangle \mathcal{M}^{-1}a^j_0 + O(\mathrm{e}^{-(2\kappa^{\mathrm{s}}+\delta)L}). \tag{5.9}$$

We next determine the form of the claimed expansions for real or complex leading stable eigenvalues. Note that $v(\alpha)$ from lemma 10 does not depend on a^1_0 or a^2_0.

If $\tilde{E}_{\kappa^{\mathrm{s}}} \neq \emptyset$ then we may choose $a^1_0 \in \tilde{E}_{\kappa^{\mathrm{s}}}$, whence $c(a^1) \neq 0$, because of (5.7), and under hypothesis 6 $c(a^1)$ is an eigenvector. We define $s_0(L) := \langle v(-2L), c(a^1)\rangle$, $b_0 := \mathcal{M}^{-1}a^1_0$ and substitution into (5.9) implies the expansion in this case. Note that $a^2_0 \notin \tilde{E}_{\kappa^{\mathrm{s}}}$ so $c(a^2_0 = 0)$.

Under hypothesis 7, $E^{\mathrm{s}}_{R^*}$ is the direct sum of complex eigenspaces to ν^{s} and $\bar{\nu}^{\mathrm{s}}$. Let w^* be an arbitrary eigenvector of R^* to ν^{s}. There are $x_j, y_j \in \mathbb{R}$ such that $c(a^j_0) = x_j\Re(w^*) + y_j\Im(w^*)$ for $j = 1,2$. If $\dim(E_{\kappa^{\mathrm{s}}}) = 2$ then $E_{\kappa^{\mathrm{s}}} = \mathrm{span}\{a^1_0, a^2_0\}$ so $c(a^j) \neq 0$ for $j = 1,2$ and we define

$$\begin{aligned}
b_1(L) &:= \sum_{j=1,2} \langle v(-2L), x_j\Re(w^*) + y_j\Im(w^*)\rangle \mathcal{M}^{-1}a^j_0\\
b_2(L) &:= \sum_{j=1,2} \langle v(-2L), y_j\Re(w^*) - y_j\Im(w^*)\rangle \mathcal{M}^{-1}a^j_0.
\end{aligned}$$

Upon substituting these into (5.9) the claimed terms in the expansion follow from a straight-forward computation using $\mathrm{e}^{R^*}\Re(w^*) = \mathrm{e}^{-\kappa^{\mathrm{s}}}(\cos(\sigma)\Re(w^*) - \sin(\sigma)\Im(w^*))$ and $\mathrm{e}^{R^*}\Im(w^*) = \mathrm{e}^{-\kappa^{\mathrm{s}}}(\sin(\sigma)\Re(w^*) + \cos(\sigma)\Im(w^*))$.

In case $\dim(E_{\kappa^{\mathrm{s}}}) = 1$ we choose $a^1_0 \in \tilde{E}_{\kappa^{\mathrm{s}}}$. Then $c(a^1) \neq 0$ and we set $s_j(L) := \langle v(-2L), (y_j - x_j)\Re(w^*) - (x_j + y_j)\Im(w^*)\rangle$ for $j = 1,2$ as well as $b_1 = b_2 :=$

$\mathcal{M}^{-1}a_0^1$. The previously mentioned computation proves the claimed expansion for this case.

Regarding periodicity, lemma 10 shows the minimal period of $v(-2L)$ in L is 0 or $mT_\gamma/(2\ell)$ for some integer $\ell \geq 1$. Therefore, the period of s_j or b_j for $j = 0$, $j = 1, 2$ respectively is as claimed.

If $\mathcal{W}_{\kappa^{\mathrm{s}}} \neq \emptyset$ and $\tilde{E}_{\kappa^{\mathrm{s}}} \neq \emptyset$ then hypothesis 5 implies that there exists $\alpha \in [0, mT_\gamma)$ such that $q_{1,\alpha}(0) \in \mathcal{W}_{\kappa^{\mathrm{s}}}$ and lemma 10 yields $v(\alpha) \neq 0$. If $s_j(L) = 0$ or $b_j(L) = 0$ then $\langle v(-2L), \Re(w^*)\rangle = 0$ and $\langle v(-2L), \Im(w^*)\rangle = 0$ because $\Re(w^*)$, $\Im(w^*)$ and a_0^1, a_0^1 are pairwise linearly independent. Conversely, if $v(\alpha) \in (\Re(E^{\mathrm{s}}_{R^*}))^\perp$ then $s_j(-\alpha/2) = 0$, $b_j(\alpha/2) = 0$ for $j = 0$ or $j = 1, 2$ respectively. □

We close this discussion with some remarks and conclusions concerning the expansions. In [26] the expansion was derived under the assumption that $\kappa^{\mathrm{s}} < \kappa^{\mathrm{u}}$ (can be relaxed to $\kappa^{\mathrm{s}} < \kappa^{\mathrm{u}} + \min\{\kappa^{\mathrm{u}}, \delta^{\mathrm{u}}\}$) for the unstable spectral gap κ^{u}. The 'flat' coordinates above were used to get rid of any restriction on the unstable spectral gap.

If $v(-2L) = 0$ for some L, then $s_j(L) = 0$ or $b_j(L) = 0$ and the heteroclinic set intersects a non-leading stable fiber of γ. We expect that an orbit flip type homoclinic bifurcation occurs, cf. e.g. [30]. Such an intersection with non-leading fibers may be structurally stable, which is not possible for heteroclinic cycles between equilibria. If $s_j(L)$ is not constant then we suspect that a countably infinite number of such bifurcations would occur.

Since $v(\alpha)$ is given in (5.5) by projections of objects with minimal period mT_γ and (5.4) contains higher order periodic corrections, it does not seem possible to conclude a nontrivial minimal period of $v(\alpha)$ in general. Concerning dimensions in which $v(\alpha)$ varies, the ambient space dimension is $n \geq 4$ and the Morse index difference is $i(p_0) - i(\gamma) = 1$. Hence we obtain $n - i(\gamma) = n + 1 - i(p_0) \geq n + 1 - (n-1) = 2$. Therefore, the strong stable fibers are at least two-dimensional.

Let τ be the minimal period of $v(-2L)$. If $\Im(2\nu^{\mathrm{s}}) \neq \tau$, then the parameter curve $\mu(L)$ 'spirals' into $\mu = 0$ along a more or less complicated path depending on the frequency ratio. However, in the one-to-one resonance the curve possibly does not spiral, and along a non-spiraling curve there are no generic saddle-nodes of the homoclinic orbits. This is in contrast to codimension–2 heteroclinic cycles between two equilibria with one transverse heteroclinic orbit ('T-points', cf. [12]). In this case, the analogue of $v(\alpha)$ is constant and spiraling is essentially equivalent to leading complex conjugate eigenvalues. In case the ODE (2.1) stems from spatial dynamics, for complex conjugate leading eigenvalues the absolute spectrum (see [32] is unstable, which typically forces infinitely many eigenvalues of the homoclinic to cross the imaginary axis as $L \to \infty$, cf. [31]. We suspect that in a non-spiraling case these eigenvalues only come in complex conjugate pairs or a condition assumed in [31] is violated.

On the other hand, for the real leading case, the vector v may be so that saddle-nodes occur despite the monotone approach in one direction in parameter space. Since this is a stable phenomenon and the absolute spectrum may be stable, we suspect that eigenvalues stabilize and destabilize periodically in this case.

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