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Analysis of bifurcations in non-smooth dynamical systems with applications to ship maneuvering

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Abstract
A general system of non-smooth equations, motivated by models for ship maneuvering, has been analyzed. In particular, a two dimensional system of ODEs with non-smoothnesses of the type $|u_i|u_j$ has been investigated. A theorem has been obtained providing the existence of a Hopf bifurcation for specific conditions for the coefficients multiplying the non-smooth terms. Furthermore, some results regarding sub- and supercriticality are presented.

Motivation and Marine Craft Modeling

A marine craft must perform certain maneuvers in order to be certified. Therefore, vessel models are examined in simulations, which involve mathematical analysis. The prediction of maneuvering properties of ships is a fundamental problem in their design.

The vectorial representation of the rigid-body equations of motion is $M\dot{\nu} = F(\nu)$, where

- ν is the velocity and $\dot{\nu}$ is its time derivative.
- M is the matrix with the added mass coefficients and the moments of inertia.
- F contains the Coriolis term and the contribution forces from the hull, the rudder and the propeller, as well as the hydrodynamic forces.

The hydrodynamic forces can be modeled employing second-order modulus terms [1], which are motivated by the drag equation for high-enough Reynolds number: $F_D = -\frac{1}{2}\rho C_D A u|u|$.

As a starting point a strongly simplified model is investigated, where products of mixed velocity components appear [2]:

$$\begin{pmatrix} m + m_{\boldsymbol{uu}} & 0 & 0 \\ 0 & m + m_{\boldsymbol{vv}} & m_{\boldsymbol{vr}} \\ 0 & m_{\boldsymbol{rv}} & I_z + m_{\boldsymbol{rr}} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{r}} \end{pmatrix} = \begin{pmatrix} m\boldsymbol{rv} + X_H + X_R + X_P \\ -m\boldsymbol{ru} + Y_H + Y_R \\ N_H + N_R \end{pmatrix}, \quad \text{where}$$

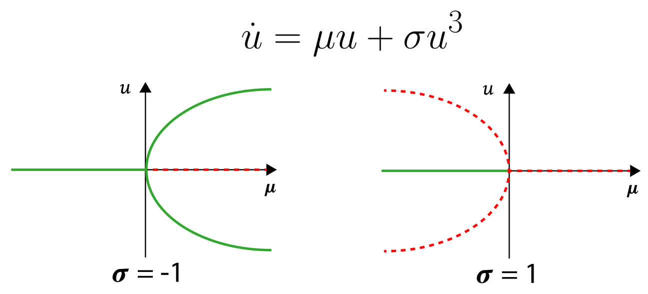
$$\begin{aligned}
X_H &= \frac{1}{2}\rho L_{pp} T \left(X'_{\boldsymbol{u}'|\boldsymbol{u}'|}\boldsymbol{u}|\boldsymbol{u}| + X'_{\beta\gamma} L_{pp} \boldsymbol{v}\,\boldsymbol{r} \right), \\
Y_H &= \frac{1}{2}\rho L_{pp} T \Big(Y'_\beta |\boldsymbol{u}|\boldsymbol{v} + Y'_\gamma L_{pp} \boldsymbol{u}\,\boldsymbol{r} + Y'_{\beta|\beta|}\boldsymbol{v}|\boldsymbol{v}| + Y'_{\gamma|\gamma|} L_{pp}^2 \boldsymbol{r}|\boldsymbol{r}| + Y'_{\beta|\gamma|} L_{pp} \boldsymbol{v}|\boldsymbol{r}| + Y'_{|\beta|\gamma} L_{pp} |\boldsymbol{v}|\boldsymbol{r} + Y'_{ab}\, |\boldsymbol{u}^{a_y}\boldsymbol{v}^{b_y}|\, \mathrm{sgn}(\boldsymbol{v}) V^{-a_y-b_y+2} \Big), \\
N_H &= \frac{1}{2}\rho L_{pp}^2 T \Big(N'_\beta \boldsymbol{u}\,\boldsymbol{v} + N'_\gamma L_{pp} \boldsymbol{r}|\boldsymbol{u}| + N'_{\boldsymbol{u}'\gamma c} L_{pp}^{c_n} |\boldsymbol{u}\,\boldsymbol{r}^{c_n}| V^{-c_n+1}\,\mathrm{sgn}(\boldsymbol{r}) + N'_{\gamma|\gamma|}\boldsymbol{r}|\boldsymbol{r}| L_{pp}^2 + N'_{\beta|\beta|}\boldsymbol{v}|\boldsymbol{v}| + N'_{\beta\beta\gamma}\boldsymbol{r}\,\boldsymbol{v}^2 L_{pp} V^{-1} \\
&\quad + N'_{\beta\gamma\gamma}\boldsymbol{v}\,\boldsymbol{r}^2 L_{pp}^2 V^{-1}\,\mathrm{sgn}(\boldsymbol{u}) + N'_{ab}|\boldsymbol{u}^{a_n}\boldsymbol{v}^{b_n}| V^{-a_n-b_n+2}\,\mathrm{sgn}(\boldsymbol{u}\,\boldsymbol{v}) \Big).
\end{aligned}$$

Inspired by this model we study a general non-smooth 2D system of ODEs, analyzing its Hopf bifurcation and the conditions for sub- and supercriticality.

Basic Bifurcation Theory

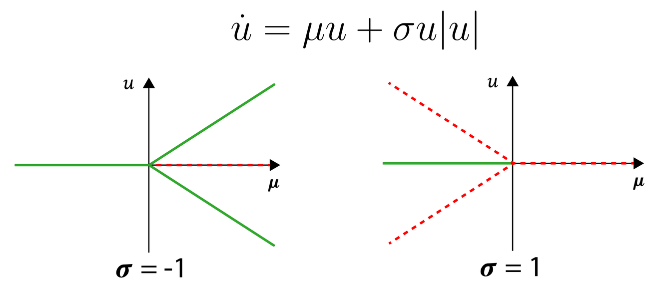
Example of the influence of the absolute value in a 1D system undergoing a pitchfork bifurcation:

$$\dot{u} = \mu u + \sigma u^3$$



Pitchfork bifurcation

$$\dot{u} = \mu u + \sigma u|u|$$



Degenerated pitchfork bifurcation

Notice that the supercriticality is given for $\sigma = -1$ while the subcriticality for $\sigma = 1$. The green lines correspond to stable points and the red dotted lines to unstable ones.

It is well known, [3], that any generic smooth 2D system undergoing a Hopf bifurcation has the following normal form in polar coordinates:

$$\begin{cases} \dot{r} &= \mu r + \sigma r^3, \\ \dot{\varphi} &= \omega, \end{cases}$$

where σ is called the first Lyapunov coefficient and μ and $\omega \neq 0$ are the parameters of the system. In the smooth case σ can be computed from a 2D vector field. Then the sub/supercriticality of the Hopf bifurcation is given by the sign of this coefficient.

For the non-smooth case the Lyapunov coefficient can not be computed in this way due to the lack of differentiability. Consequently, new results have been derived in order to identify such bifurcations.

Theorem *(1st part):*

Consider the following system:

$$\begin{cases} \dot{u} &= \mu u - \omega v - f\left(u, |u|, v, |v|\right), \\ \dot{v} &= \omega u + \mu v - g\left(u, |u|, v, |v|\right), \end{cases} \tag{1}$$

where

$$f\left(u, |u|, v, |v|\right) = a_{11}u|u| + a_{12}u|v| + a_{21}v|u| + a_{22}v|v|,$$
$$g\left(u, |u|, v, |v|\right) = b_{11}u|u| + b_{12}u|v| + b_{21}v|u| + b_{22}v|v|.$$

If $\sigma_{\#} := 2a_{11} + a_{12} + b_{21} + 2b_{22} \neq 0$, then there exists a Hopf bifurcation in μ where the amplitude of the locally unique limit cycle is

$$r = \frac{3\pi}{2\sigma_{\#}}\mu + \mathcal{O}\left(\mu^2\right),$$

where $u = r\cos\varphi$, $v = r\sin\varphi$. The Hopf bifurcation is then subcritical if $\mathrm{sgn}(\sigma_{\#}) < 0$ and supercritical if $\mathrm{sgn}(\sigma_{\#}) > 0$.

Let us compare $\sigma_{\#}$ with the first Lyapunov coefficient of a smooth version of (1) with the same symmetry by setting $f\left(u, |u|, v, |v|\right) \mapsto f\left(u, u^2, v, v^2\right)$ and $g\left(u, |u|, v, |v|\right) \mapsto g\left(u, u^2, v, v^2\right)$:

$$\sigma_{\mathtt{smooth}} = \frac{1}{4\omega}\left(3a_{11} + a_{12} + b_{21} + 3b_{22}\right).$$

There is a qualitatively distinction between both coefficients (not just a scalar variation) and generally $\mathrm{sgn}(\sigma_{\#}) \neq \mathrm{sgn}(\sigma_{\mathtt{smooth}})$. Whence, the sub/supercriticality can not simply be generalized from the smooth case in this way.

Theorem *(2nd part):*

If $\sigma_{\#} = 0$ but $\alpha := \sum\left(c_{ik}a_{ij}b_{kh} + d_{ik}a_{ij}a_{kh} + s_{ik}b_{ij}b_{kh}\right) \neq 0$, with c_{ik}, d_{ik}, s_{ik} certain constants, then there also occurs a bifurcation of a periodic solution in μ where the amplitude of the unique limit cycle follows

$$r = \sqrt{-\frac{\pi\omega}{\alpha}\mu} + \mathcal{O}\left(\mu\right).$$

The sub/supercriticality is given by $\mathrm{sgn}(\alpha)$.

One could also compare α with the second Lyapunov coefficient [4].
Note that the periodic branch is $\mathcal{O}\left(1/2\right)$ while in the smooth case is $\mathcal{O}\left(1/4\right)$.

Theorem *(3rd part):*

If $\sigma_{\#} \neq 0$, then the same results hold for the more general system with quadratic and cubic terms

$$\begin{cases} \dot{u} &= \mu u - \omega v - f\left(u, |u|, v, |v|\right) + f_q\left(u, v\right) + f_c\left(u, u^2, v, v^2\right), \\ \dot{v} &= \omega u + \mu v - g\left(u, |u|, v, |v|\right) + g_q\left(u, v\right) + g_c\left(u, u^2, v, v^2\right). \end{cases}$$

In particular, f_q, g_q, f_c and g_c do not change $\sigma_{\#}$.

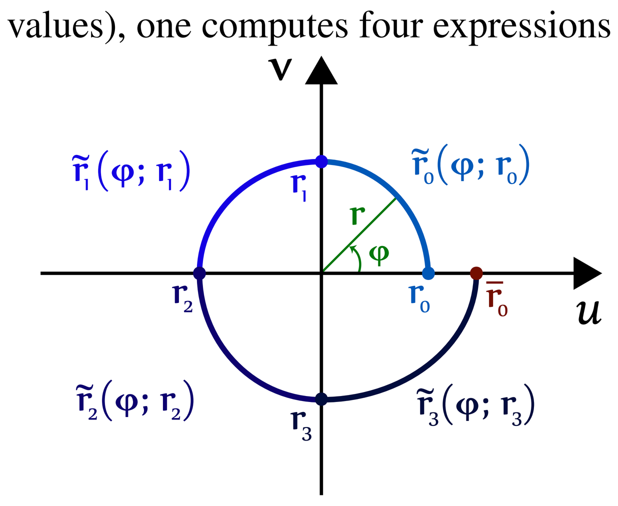
Poincaré Map

We use polar coordinates and introduce a new time parametrization to the system (1). We also perform the Taylor expansion to get the following system:

$$\begin{cases} r' := \frac{dr}{d\varphi} &= \frac{\mu r - r^2 F(\varphi)}{\omega + r\Omega(\varphi)} = \frac{\mu}{\omega}r - r^2\left(\frac{F(\varphi)}{\omega} + \frac{\mu\Omega(\varphi)}{\omega^2}\right) + \mathcal{O}(r^3), \\ \varphi' := \frac{d\varphi}{d\varphi} &= 1, \end{cases} \tag{2}$$

where $F(\varphi) = F(\cos\varphi, |\cos\varphi|, \sin\varphi, |\sin\varphi|)$ and $\Omega(\varphi) = \Omega(\cos\varphi, |\cos\varphi|, \sin\varphi, |\sin\varphi|)$.

We study the full dynamics by computing the Poincaré map. Since the system is non-smooth (absolute values), one computes four expressions (one for each quadrant), as it is shown in the following picture.



Afterwards one composes these expressions to get the Poincaré map $\tilde{r}_3\left(2\pi; r_0\right)$.
The Poincaré section is defined as $S = \{(u, 0) \mid u \in \mathbb{R}_{>0}\}$ and the initial condition is $(\varphi, r) = (0, r_0)$.
We write then

$$\tilde{r}_i(\varphi; r_i) = \alpha_{1,i}(\varphi)r_i + \alpha_{2,i}(\varphi)r_i^2 + \mathcal{O}(r_i^3) \tag{3}$$

since $\tilde{r}_i(\varphi; 0) = 0\ \forall i \in \{0, \ldots, 3\}$ holds.

With the corresponding initial conditions one computes $\alpha_{1,i}(\varphi)$ and $\alpha_{2,i}(\varphi)$ substituting (3) in r' of (2), and solving the resulting expression. Finally, one calculates r_3 in terms of r_0. Then, the next iteration of r_0 is given by $\tilde{r}_3\left(2\pi; r_0\right) = \bar{r_0}$.

The periodic orbit of the original system (1) corresponds to the smallest positive solution (due to the leading order term) of

$$r^* = \tilde{r}_3\left(2\pi; r^*\right),$$

which is approximately the one obtained by the theorem. The sub/supercriticality is investigated as always, which certainly coincides with the result of the theorem.

Forthcoming Research

- ⋄ Theorem in 3D: the variables of the initial model (1) will be related to degrees of freedom of a vessel and the system will be extended to a 3D model in order to compare results with [2].
- ⋄ To consider certain maneuvers as boundary value problems.
- ⋄ To add control (multi-objective optimization). ⟵ for the straight motion there exist already results for stability

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