A general system of non-smooth equations, motivated by models for ship maneuvering, has been analyzed. In particular, a two-dimensional system of ODEs with a non-smoothness of the type $\mu |x|$, has been investigated. A theorem has been obtained providing the existence of a Hopf bifurcation for specific conditions for the coefficients multiplying the non-smooth terms. Furthermore, some results regarding sub- and supercriticality are presented.

**Motivation and Marine Craft Modeling**

A marine craft must perform certain maneuvers in order to be certified. Therefore, vessel models are examined in simulations, which involve mathematical analysis. The prediction of maneuvering properties of ships is a fundamental problem in their design.

The vectorial representation of the rigid-body equations of motion is $M\ddot{\mathbf{v}} = F(\mathbf{v})$, where
- $\mathbf{v}$ is the velocity and $\dot{\mathbf{v}}$ is its time derivative.
- $M$ is the matrix with the added mass and the second-order moments of inertia.
- $F$ contains the Coriolis term and the contribution forces from the ball, the rudder and the propeller, as well as the hydrodynamical forces.

The hydrodynamical forces can be modeled employing second-order terms [3], which are motivated by the drag equation for high-enough Reynolds number $F_D = \frac{1}{2} \rho L \mu |\mathbf{v}|^2$. 

For the non-smooth case the Lyapunov coefficient cannot be computed in this way due to the lack of differentiability. Consequently, new results have been derived in order to identify such bifurcations. For the smooth case the Lyapunov coefficient is given by the sign of this coefficient.

Let us compare $\sigma_j$ with the first Lyapunov coefficient of a smooth version of [3], with the same symmetry by setting $f(u, |u|, v, |v|) \rightarrow f(u, u^2, v, v^2)$ and $g(u, |u|, v, |v|) \rightarrow g(u, u^2, v, v^2)$:

$$\sigma_{\text{smooth}}^{(1)} = \frac{1}{2} \rho L (\alpha_1 + \alpha_2 + \beta_1 + \beta_2).$$

There is a qualitatively distinction between both coefficients (not just a scalar variation) and generally $\sigma_j(\sigma_j) \neq \sigma_j(\text{smooth})$. Whence, the sub/supercriticality cannot simply be generalized from the smooth case in this way.

**Theorem (2nd part):**

If $\sigma_j = 0$ but $\alpha = \sum \{c_{ij}n_i, a_{ij}, b_{ij} | a_{ij}, b_{ij} | \neq 0 \}$, with $c_{ij}, a_{ij}, b_{ij}$ certain constants, then there also occurs a bifurcation of a periodic solution in $\omega$ where the amplitude of the unique limit cycle follows

$$r = \sqrt{\frac{\alpha^2}{\omega}} + O(\omega).$$

The sub/supercriticality is given by $\sigma(\omega) = 0$. One could also compare $\alpha$ with the second Lyapunov coefficient $\beta_j$.

Note that the periodic branch is $O(1/\omega)$ while in the smooth case is $O(1)$. 

**Theorem (3rd part):**

If $\sigma_j(\sigma_j) \neq 0$, then the same results hold for the more general system with quadratic and cubic terms

$$\begin{align*}
\dot{u} &= \mu u - \omega v - f(u, |u|, v, |v|) + f_u(u, v) + f_v(u, v^2), \\
\dot{v} &= \omega u + \mu v - g(u, |u|, v, |v|) + g_u(u, v) + g_v(u, v^2, v^3).
\end{align*}$$

In particular, $f_u, g_u, f_v$ and $g_v$ do not change $\sigma_j$.

**Poincaré Map**

We use polar coordinates and introduce a new time parametrization to the system [3]. We also perform the Taylor expansion to get the following system:

$$\begin{align*}
\dot{\rho} &= \frac{\sigma_j(\rho^2) + \sigma_{\text{smooth}}^{(1)}(\rho^2) + \sigma_{\text{smooth}}^{(2)}(\rho^2)}{\sigma_{\text{smooth}}^{(1)}}, \\
\dot{\tau} &= \frac{1}{1},
\end{align*}$$

where $F(\rho) = \rho \cos \omega \tau, \rho \sin \omega \tau, \rho \sin \omega \tau$ and $\Omega(\rho, \rho \sin \omega \tau, \rho \sin \omega \tau)$. 

We study the full dynamics by computing the Poincaré map. Since the system is non-smooth (absolute values), one computes four expressions (one for each quadrant), as it is shown in the following picture. Afterwords one compares these expressions to get the Poincaré section.

The Poincaré section is defined as $S = \{ (\omega, r) \mid r \in [0, \infty) \}$ and the initial condition is $(0, r_0)$. We write then

$$f(\rho, \rho \cos \omega \tau, \rho \sin \omega \tau, \rho \sin \omega \tau) = \Omega(\rho, \rho \sin \omega \tau, \rho \sin \omega \tau).$$

Solving the resulting expression. Finally, one calculates $r_0$ in terms of $r_0$. Then, the next iteration of $r_0$ is given by $f(\rho, \rho \cos \omega \tau, \rho \sin \omega \tau, \rho \sin \omega \tau)$. 

The periodic orbit of the original system [3] corresponds to the smallest positive solution (due to the leading order term) of

$$r^* = f(\rho, \rho \cos \omega \tau, \rho \sin \omega \tau),$$

which is approximately the one obtained by the theorem. The sub/supercriticality is investigated as always, which certainly coincides with the result of the theorem.

**Forthcoming Research**

- Theory in 3D the variables of the initial model [3] will be related to degrees of freedom of a vessel and the system will be extended to a 3D model in order to compare results with [2].
- To consider certain maneuver as boundary value problems.
- To add control (multi-objective optimization).

References